Stochastic equivalence for modular performance evaluation in discrete time stochastic Petri box calculus

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Abstract: Algebra $dtsPBC$ is a discrete time stochastic extension of finite Petri box calculus (PBC) enriched with iteration.

The step operational semantics is defined in terms of labeled probabilistic transition systems.

The denotational semantics is defined in terms of a subclass of labeled DTSPNs (LDTSPNs) called discrete time stochastic Petri boxes (dts-boxes).

We propose and investigate step stochastic bisimulation equivalence.

This equivalence is used for the reduction of transition systems and Markov chains.

The mentioned equivalence is applied to compare stationary behaviour.

A method of modeling and performance evaluation based on stationary behaviour analysis and reduction for concurrent systems is outlined applied to the shared memory system.

Keywords: stochastic Petri net, stochastic process algebra, Petri box calculus, iteration, discrete time, transition system, operational semantics, dts-box, denotational semantics, Markov chain, performance evaluation, stochastic equivalence, reduction, shared memory system.

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Introduction

Algebra PBC and its extensions

- *Petri box calculus* PBC [BDH92]
- *Time Petri box calculus* $tPBC$ [Kou00]
- *Timed Petri box calculus* $TPBC$ [MF00]
- *Stochastic Petri box calculus* $sPBC$ [MVF01,MVCC03]
- *Ambient Petri box calculus* $APBC$ [FM03]
- *Arc time Petri box calculus* $atPBC$ [Nia05]
- *Generalized stochastic Petri box calculus* $gsPBC$ [MVCR08]
- *Discrete time stochastic Petri box calculus* $dtsPBC$ [Tar05,Tar06]
- *Discrete time stochastic and immediate Petri box calculus* $dtsiPBC$ [TMV10]

Syntax

The *set of all finite multisets* over X is $\mathbb{N}_f^X$.

The *set of all subsets* of X is 2^X.

$Act = \{a, b, \ldots\}$ is the set of *elementary actions*.

$\widehat{Act} = \{\hat{a}, \hat{b}, \ldots\}$ is the set of *conjugated actions (conjugates)* s.t. $a \neq \hat{a}$ and $\hat{\hat{a}} = a$.

$\mathcal{A} = Act \cup \widehat{Act}$ is the set of *all actions*.

$\mathcal{L} = \mathbb{N}_f^{\mathcal{A}}$ is the set of *all multiactions*.

The *alphabet* of $\alpha \in \mathcal{L}$ is $\mathcal{A}(\alpha) = \{x \in \mathcal{A} \mid \alpha(x) > 0\}$.

An *activity (stochastic multiaction)* is a pair (α, ρ), where
$\alpha \in \mathcal{L}$ and $\rho \in (0; 1)$ is the *probability* of multiaction α.

$\mathcal{SL}$ is the set of *all activities*.

The *alphabet* of $(\alpha, \rho) \in \mathcal{SL}$ is $\mathcal{A}(\alpha, \rho) = \mathcal{A}(\alpha)$.

The *alphabet* of $\Gamma \in \mathbb{N}_f^{\mathcal{SL}}$ is $\mathcal{A}(\Gamma) = \cup_{(\alpha,\rho)\in\Gamma}\mathcal{A}(\alpha)$.

For $(\alpha, \rho) \in \mathcal{SL}$, its *multiaction part* is $\mathcal{L}(\alpha, \rho) = \alpha$ and its *probability part* is $\Omega(\alpha, \rho) = \rho$.

The *multiaction part* of $\Gamma \in \mathbb{N}_f^{\mathcal{SL}}$ is $\mathcal{L}(\Gamma) = \sum_{(\alpha,\rho)\in\Gamma} \alpha$.

The operations: *sequential execution* $;$, *choice* $[]$, *parallelism* $\|$, *relabeling* $[f]$, *restriction* rs, *synchronization* sy and *iteration* $[**]$.

Sequential execution and choice have the standard interpretation.

Parallelism does not include synchronization unlike that in standard process algebras.

Relabeling functions $f : \mathcal{A} \to \mathcal{A}$ are bijections preserving conjugates: $\forall x \in \mathcal{A}\ f(\hat{x}) = \widehat{f(x)}$.
For $\alpha \in \mathcal{L}$, let $f(\alpha) = \sum_{x\in\alpha} f(x)$. For $\Gamma \in I\!N_f^{\mathcal{SL}}$, let $f(\Gamma) = \sum_{(\alpha,\rho)\in\Gamma}(f(\alpha), \rho)$.

Restriction over an action a: any process behaviour containing a or its conjugate $\hat{a}$ is not allowed.

Let $\alpha, \beta \in \mathcal{L}$ be two multiactions s.t. for $a \in Act$ we have $a \in \alpha$ and $\hat{a} \in \beta$ or $\hat{a} \in \alpha$ and $a \in \beta$.
Synchronization of α and β by a is $\alpha \oplus_a \beta = \gamma$:

$$\gamma(x) = \begin{cases} \alpha(x) + \beta(x) - 1, & x = a \text{ or } x = \hat{a}; \\ \alpha(x) + \beta(x), & \text{otherwise}. \end{cases}$$

In the iteration, the initialization subprocess is executed first,
then the body one is performed zero or more times, finally, the termination one is executed.

Static expressions specify the structure of processes.

Definition 1 *Let* $(\alpha, \rho) \in \mathcal{SL}$ *and* $a \in Act$. *A* static expression *of* $dtsPBC$ *is*

$$E ::= \ (\alpha, \rho) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}E{*}E].$$

$StatExpr$ is the set of *all static expressions* of $dtsPBC$.

Definition 2 *Let* $(\alpha, \rho) \in \mathcal{SL}$ *and* $a \in Act$. *A* regular static expression *of* $dtsPBC$ *is*

$$E ::= \ (\alpha, \rho) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}D{*}E],$$

where $D ::= \ (\alpha, \rho) \mid D;E \mid D[]D \mid D[f] \mid D \text{ rs } a \mid D \text{ sy } a \mid [D{*}D{*}E]$.

$RegStatExpr$ is the set of *all regular static expressions* of $dtsPBC$.

Dynamic expressions specify the states of processes.

Dynamic expressions are combined from static ones annotated with upper or lower bars.

The *underlying static expression* of a dynamic one: removing all upper and lower bars.

Definition 3 *Let $E \in StatExpr$ and $a \in Act$. A dynamic expression of $dtsPBC$ is*

$$G ::= \overline{E} \mid \underline{E} \mid G;E \mid E;G \mid G[]E \mid E[]G \mid G\|G \mid G[f] \mid G \text{ rs } a \mid G \text{ sy } a \mid [G*E*E] \mid [E*G*E] \mid [E*E*G].$$

$DynExpr$ is the set of *all dynamic expressions* of $dtsPBC$.

A *regular dynamic expression*: its underlying static expression is regular.

$RegDynExpr$ is the set of *all regular dynamic expressions* of $dtsPBC$.

Operational semantics

Inaction rules

Inaction rules: instantaneous structural transformations.

Let $E, F, K \in RegStatExpr$ and $a \in Act$.

Inaction rules for overlined and underlined regular static expressions

$\overline{E;F} \Rightarrow \overline{E};F$	$\underline{E};F \Rightarrow E;\overline{F}$	$E;\underline{F} \Rightarrow \underline{E;F}$
$\overline{E[]F} \Rightarrow \overline{E}[]F$	$\overline{E[]F} \Rightarrow E[]\overline{F}$	$\underline{E}[]F \Rightarrow \underline{E[]F}$
$E[]\underline{F} \Rightarrow \underline{E[]F}$	$\overline{E\|F} \Rightarrow \overline{E}\|\overline{F}$	$\underline{E}\|\underline{F} \Rightarrow \underline{E\|F}$
$\overline{E[f]} \Rightarrow \overline{E}[f]$	$\underline{E}[f] \Rightarrow \underline{E[f]}$	$\overline{E \text{ rs } a} \Rightarrow \overline{E} \text{ rs } a$
$\underline{E} \text{ rs } a \Rightarrow \underline{E \text{ rs } a}$	$\overline{E \text{ sy } a} \Rightarrow \overline{E} \text{ sy } a$	$\underline{E} \text{ sy } a \Rightarrow \underline{E \text{ sy } a}$
$\overline{[E*F*K]} \Rightarrow [\overline{E}*F*K]$	$[\underline{E}*F*K] \Rightarrow [E*\overline{F}*K]$	$[E*\underline{F}*K] \Rightarrow [E*\overline{F}*K]$
$[E*\underline{F}*K] \Rightarrow [E*F*\overline{K}]$	$[E*F*\underline{K}] \Rightarrow \underline{[E*F*K]}$	

Let $E, F \in RegStatExpr,\ G, H, \widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$.

Inaction rules for arbitrary regular dynamic expressions

$$\frac{G\Rightarrow\widetilde{G},\ \circ\in\{;,[]\}}{G\circ E\Rightarrow\widetilde{G}\circ E} \quad \frac{G\Rightarrow\widetilde{G},\ \circ\in\{;,[]\}}{E\circ G\Rightarrow E\circ\widetilde{G}} \quad \frac{G\Rightarrow\widetilde{G}}{G\|H\Rightarrow\widetilde{G}\|H} \quad \frac{H\Rightarrow\widetilde{H}}{G\|H\Rightarrow G\|\widetilde{H}} \quad \frac{G\Rightarrow\widetilde{G}}{G[f]\Rightarrow\widetilde{G}[f]}$$

$$\frac{G\Rightarrow\widetilde{G},\ \circ\in\{\text{rs},\text{sy}\}}{G\circ a\Rightarrow\widetilde{G}\circ a} \quad \frac{G\Rightarrow\widetilde{G}}{[G*E*F]\Rightarrow[\widetilde{G}*E*F]} \quad \frac{G\Rightarrow\widetilde{G}}{[E*G*F]\Rightarrow[E*\widetilde{G}*F]} \quad \frac{G\Rightarrow\widetilde{G}}{[E*F*G]\Rightarrow[E*F*\widetilde{G}]}$$

An *operative regular dynamic expression* G: no inaction rule can be applied to it.

$OpRegDynExpr$ is the set of *all operative regular dynamic expressions* of $dtsPBC$.

We shall consider regular expressions only and omit the word “regular”.

Definition **4** *$\approx\ = (\Rightarrow \cup \Leftarrow)^*$ is the structural equivalence of dynamic expressions in $dtsPBC$. G and G' are structurally equivalent, $G\approx G'$, if they can be reached each from other by applying inaction rules in forward or backward direction.*

Action and empty loop rules

Action rules: execution of non-empty multisets of activities at a time step.

Empty loop rule: execution of the empty multiset of activities at a time step.

Let $(\alpha,\rho),(\beta,\chi)\in\mathcal{SL},\ E,F\in RegStatExpr,\ G,H\in OpRegDynExpr,$ $\widetilde{G},\widetilde{H}\in RegDynExpr,\ a\in Act$ and $\Gamma,\Delta\in\mathbb{N}_f^{\mathcal{SL}}\setminus\{\emptyset\},\ \Gamma'\in\mathbb{N}_f^{\mathcal{SL}}$.

Action and empty loop rules

$$\textbf{El}\ G\stackrel{\emptyset}{\rightarrow}G \qquad \textbf{B}\ \overline{(\alpha,\rho)}\stackrel{\{(\alpha,\rho)\}}{\longrightarrow}\underline{(\alpha,\rho)} \qquad \textbf{SC1}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ \circ\in\{;,[]\}}{G\circ E\stackrel{\Gamma}{\rightarrow}\widetilde{G}\circ E}$$

$$\textbf{SC2}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ \circ\in\{;,[]\}}{E\circ G\stackrel{\Gamma}{\rightarrow}E\circ\widetilde{G}} \qquad \textbf{P1}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{G\|H\stackrel{\Gamma}{\rightarrow}\widetilde{G}\|H} \qquad \textbf{P2}\ \frac{H\stackrel{\Gamma}{\rightarrow}\widetilde{H}}{G\|H\stackrel{\Gamma}{\rightarrow}G\|\widetilde{H}}$$

$$\textbf{P3}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ H\stackrel{\Delta}{\rightarrow}\widetilde{H}}{G\|H\stackrel{\Gamma+\Delta}{\longrightarrow}\widetilde{G}\|\widetilde{H}} \qquad \textbf{L}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{G[f]\stackrel{f(\Gamma)}{\longrightarrow}\widetilde{G}[f]} \qquad \textbf{Rs}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ a,\hat{a}\not\in\mathcal{A}(\Gamma)}{G\ \mathsf{rs}\ a\stackrel{\Gamma}{\rightarrow}\widetilde{G}\ \mathsf{rs}\ a}$$

$$\textbf{I1}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{[G*E*F]\stackrel{\Gamma}{\rightarrow}[\widetilde{G}*E*F]} \qquad \textbf{I2}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{[E*G*F]\stackrel{\Gamma}{\rightarrow}[E*\widetilde{G}*F]} \qquad \textbf{I3}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{[E*F*G]\stackrel{\Gamma}{\rightarrow}[E*F*\widetilde{G}]}$$

$$\textbf{Sy1}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{G\ \mathsf{sy}\ a\stackrel{\Gamma}{\rightarrow}\widetilde{G}\ \mathsf{sy}\ a} \qquad \textbf{Sy2}\ \frac{G\ \mathsf{sy}\ a\xrightarrow{\Gamma'+\{(\alpha,\rho)\}+\{(\beta,\chi)\}}\widetilde{G}\ \mathsf{sy}\ a,\ a\in\alpha,\ \hat{a}\in\beta}{G\ \mathsf{sy}\ a\xrightarrow{\Gamma'+\{(\alpha\oplus_a\beta,\rho\cdot\chi)\}}\widetilde{G}\ \mathsf{sy}\ a}$$

Transition systems

Definition 5 *Let $n \in I\!N$. The numbering of expressions is $\iota ::= \ n \mid (\iota)(\iota)$.*

Num is the set of *all numberings* of expressions.

The *content* of a numbering $\iota \in Num$ is

$$Cont(\iota) = \begin{cases} \{\iota\}, & \iota \in I\!N; \\ Cont(\iota_1) \cup Cont(\iota_2), & \iota = (\iota_1)(\iota_2). \end{cases}$$

$[G]_\approx = \{H \mid G \approx H\}$ is the equivalence class of $G \in RegDynExpr$ w.r.t. structural equivalence.

Definition 6 *The derivation set $DR(G)$ of a dynamic expression G is the minimal set:*

- $[G]_\approx \in DR(G)$;
- *if* $[H]_\approx \in DR(G)$ *and* $\exists \Gamma \ H \xrightarrow{\Gamma} \widetilde{H}$ *then* $[\widetilde{H}]_\approx \in DR(G)$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G)$.

The set of *all multisets of activities executable from s* is $Exec(s) = \{\Gamma \mid \exists H \in s \ \exists \widetilde{H} \ H \xrightarrow{\Gamma} \widetilde{H}\}$.

Let $\Gamma \in Exec(s) \setminus \{\emptyset\}$. The *probability that the multiset of activities Γ is ready for execution in s*:

$$PF(\Gamma, s) = \prod_{(\alpha,\rho)\in\Gamma} \rho \cdot \prod_{\{\{(\beta,\chi)\}\in Exec(s) \mid (\beta,\chi)\notin\Gamma\}} (1-\chi).$$

In the case $\Gamma = \emptyset$ we define $PF(\emptyset, s) = \begin{cases} \prod_{\{(\beta,\chi)\}\in Exec(s)} (1-\chi), & Exec(s) \neq \{\emptyset\}; \\ 1, & \text{otherwise}. \end{cases}$

Let $\Gamma \in Exec(s)$. The *probability to execute the multiset of activities Γ in s*:

$$PT(\Gamma, s) = \frac{PF(\Gamma, s)}{\sum_{\Delta\in Exec(s)} PF(\Delta, s)}.$$

The *probability to move from s to $\tilde{s}$ by executing any multiset of activities*:

$$PM(s, \tilde{s}) = \sum_{\{\Gamma \mid \exists H\in s\ \exists \widetilde{H}\in\tilde{s}\ H \xrightarrow{\Gamma} \widetilde{H}\}} PT(\Gamma, s).$$

Definition 7 *The (labeled probabilistic) transition system of a dynamic expression G is* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, *where*

- *the set of states is* $S_G = DR(G)$;
- *the set of labels is* $L_G \subseteq \mathbb{N}_f^{\mathcal{SL}} \times (0;1]$;
- *the set of transitions is* $\mathcal{T}_G = \{(s, (\Gamma, PT(\Gamma, s)), \tilde{s}) \mid s \in DR(G),\ \exists H \in s\, \exists \widetilde{H} \in \tilde{s}\ H \xrightarrow{\Gamma} \widetilde{H}\}$;
- *the initial state is* $s_G = [G]_\approx$.

A transition $(s, (\Gamma, \mathcal{P}), \tilde{s}) \in \mathcal{T}_G$ is written as $s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s}$.

We write $s \xrightarrow{\Gamma} \tilde{s}$ if $\exists \mathcal{P}\ s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s}$ and $s \rightarrow \tilde{s}$ if $\exists \Gamma\ s \xrightarrow{\Gamma} \tilde{s}$.

Definition 8 *Let G, G' be dynamic expressions and $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, $TS(G') = (S_{G'}, L_{G'}, \mathcal{T}_{G'}, s_{G'})$ be their transition systems. A mapping $\beta : S_G \rightarrow S_{G'}$ is an* isomorphism *between $TS(G)$ and $TS(G')$, $\beta : TS(G) \simeq TS(G')$, if*

1. β *is a bijection s.t.* $\beta(s_G) = s_{G'}$;
2. $\forall s, \tilde{s} \in S_G\ \forall \Gamma\ s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s} \Leftrightarrow \beta(s) \xrightarrow{\Gamma}_{\mathcal{P}} \beta(\tilde{s})$.

$TS(G)$ *and* $TS(G')$ *are* isomorphic, $TS(G) \simeq TS(G')$, *if* $\exists \beta : TS(G) \simeq TS(G')$.

For $E \in RegStatExpr$, let $TS(E) = TS(\overline{E})$.

Definition 9 *G and G' are equivalent w.r.t. transition systems, $G =_{ts} G'$, if $TS(G) \simeq TS(G')$.*

Definition 10 *The underlying discrete time Markov chain (DTMC) of a dynamic expression G, $DTMC(G)$, has the state space $DR(G)$ and transitions $s \rightarrow_{\mathcal{P}} \tilde{s}$, if $s \rightarrow \tilde{s}$ and $\mathcal{P} = PM(s, \tilde{s})$.*

For $E \in RegStatExpr$, let $DTMC(E) = DTMC(\overline{E})$.

For a dynamic expression G, a discrete random variable is associated with every state of $DTMC(G)$.

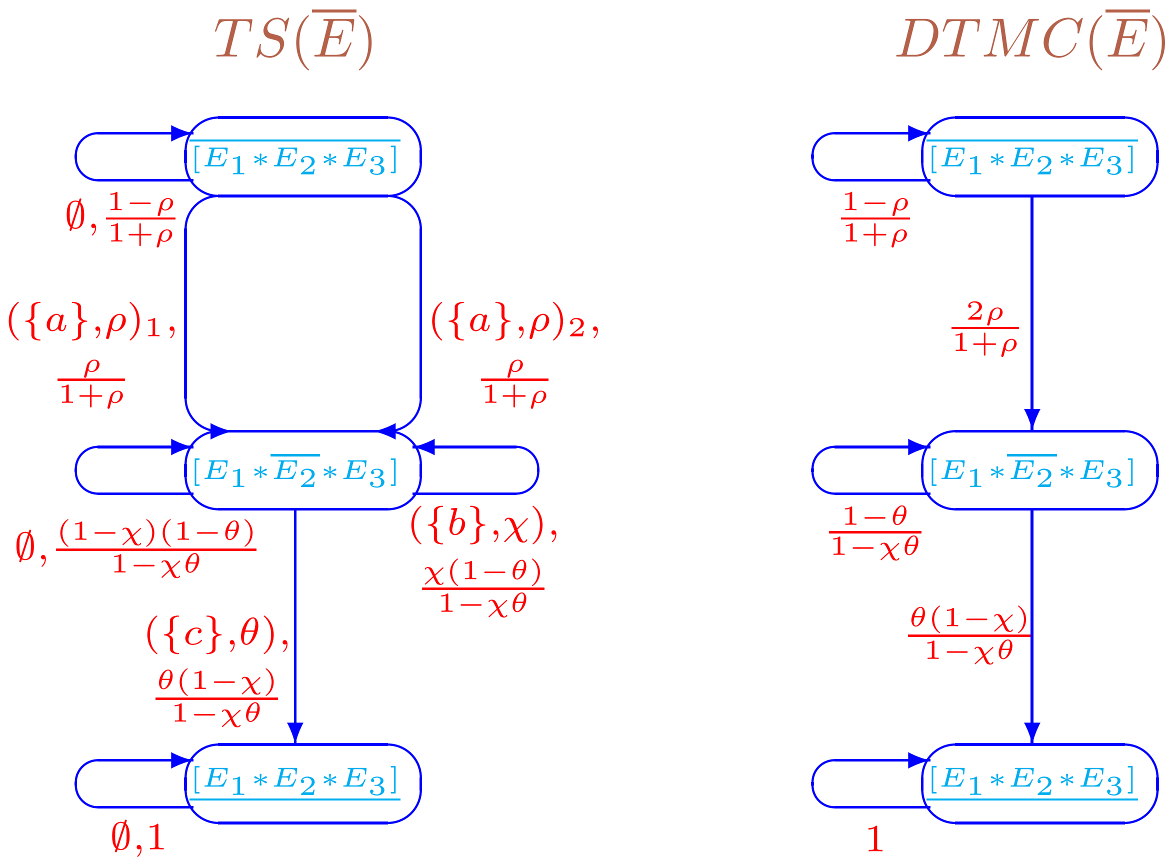
The random values (residence time in the states) are geometrically distributed:
the probability to stay in the state $s \in DR(G)$ for $k-1$ moments and leave it at moment $k \geq 1$ is $PM(s,s)^{k-1}(1 - PM(s,s))$.

The mean value formula: the *average sojourn time in the state s* is $SJ(s) = \frac{1}{1-PM(s,s)}$.

The *average sojourn time vector* SJ of G is that with the elements $SJ(s),\ s \in DR(G)$.

Analogously: the *sojourn time variance in the state s* is $VAR(s) = \frac{PM(s,s)}{(1-PM(s,s))^2}$.

The *sojourn time variance vector* VAR of G is that with the elements $VAR(s),\ s \in DR(G)$.



EXPRIT:The transition system and the underlying DTMC of $\overline{E}$ for $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$

Let $E_1 = (\{a\},\rho)[](\{a\},\rho),\ E_2 = (\{b\},\chi),\ E_3 = (\{c\},\theta)$ and $E = [E_1 * E_2 * E_3]$.

The identical activities of the composite static expression are enumerated as:

$E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$. The derivation set $DR(\overline{E})$ of $\overline{E}$ consists of

$s_1 = [[\overline{E_1 * E_2 * E_3}]]_\approx,\ s_2 = [[E_1 * \overline{E_2} * E_3]]_\approx,\ s_3 = [[\underline{E_1 * E_2 * E_3}]]_\approx$.

The average sojourn time vector of $\overline{E}$ is

$$SJ = \left(\frac{1+\rho}{2\rho}, \frac{1-\chi\theta}{\theta(1-\chi)}, \infty\right).$$

The sojourn time variance vector of $\overline{E}$ is

$$VAR = \left(\frac{1-\rho^2}{4\rho^2}, \frac{(1-\theta)(1-\chi\theta)}{\theta^2(1-\chi)^2}, \infty\right).$$

Algebra of dts-boxes

Definition 11 *A* discrete time stochastic Petri box (dts-box) *is* $N = (P_N, T_N, W_N, \Lambda_N)$*, where*

- P_N *and* T_N *are finite sets of* places *and* transitions*, respectively, s.t.* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$*;*
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to I\!N$ *is a function of the* weights of arcs *between places and transitions and vice versa;*
- Λ_N *is the* place and transition labeling *function s.t.*
  - $\Lambda_N|_{P_N} : P_N \to \{\mathsf{e}, \mathsf{i}, \mathsf{x}\}$ *(it specifies* entry, internal *and* exit *places);*
  - $\Lambda_N|_{T_N} : T_N \to \{\varrho \mid \varrho \subseteq I\!N_f^{\mathcal{SL}} \times \mathcal{SL}\}$ *(it associates transitions with the* relabeling relations*).*

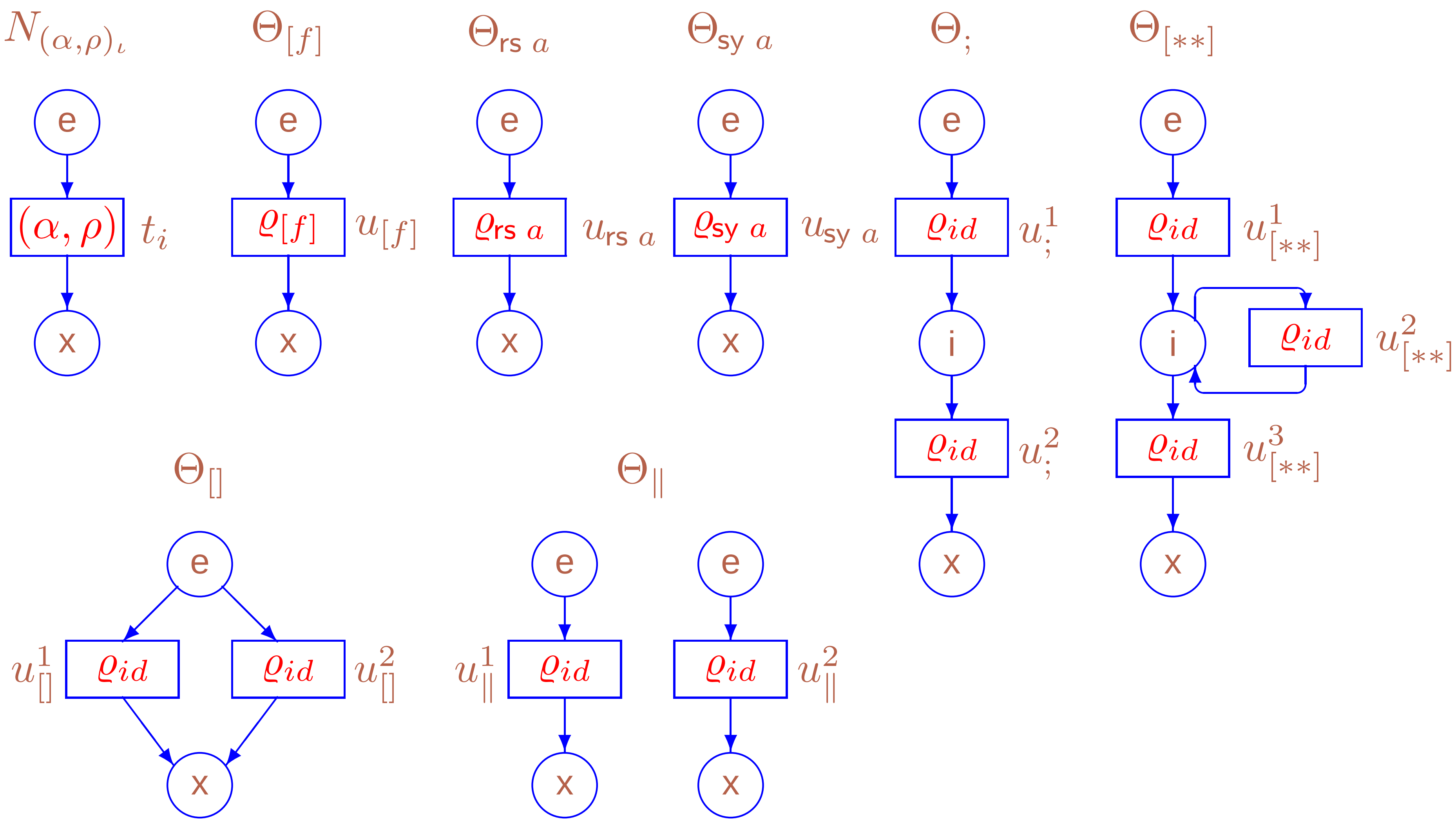
Moreover, $\forall t \in T_N\ {}^\bullet t \neq \emptyset \neq t^\bullet$*.*

For the set of entry *places of* N*,* ${}^\circ N = \{p \in P_N \mid \Lambda_N(p) = \mathsf{e}\}$*, and the set of* exit *places of* N*,* $N^\circ = \{p \in P_N \mid \Lambda_N(p) = \mathsf{x}\}$*, it holds:* ${}^\circ N \neq \emptyset \neq N^\circ$ *and* ${}^\bullet({}^\circ N) = \emptyset = (N^\circ)^\bullet$*.*

A dts-box is *plain* if $\forall t \in T_N\ \Lambda_N(t) \in \mathcal{SL}$, i.e., $\Lambda_N(t)$ is the constant relabeling.

A *marked plain dts-box* is a pair (N, M_N), where N is a plain dts-box and $M_N \in I\!N_f^{P_N}$ is its marking. Let $\overline{N} = (N, {}^\circ N)$ and $\underline{N} = (N, N^\circ)$.

Denotational semantics



The plain and operator dts-boxes

Definition 12 *Let* $(\alpha, \rho) \in \mathcal{SL}$, $a \in Act$ *and* $E, F, K \in RegStatExpr$. *The* denotational semantics *of* $dtsPBC$ *is a mapping* Box_{dts} *from* $RegStatExpr$ *into plain dts-boxes:*

1. $Box_{dts}((\alpha, \rho)_\iota) = N_{(\alpha,\rho)_\iota}$;
2. $Box_{dts}(E \circ F) = \Theta_\circ(Box_{dts}(E), Box_{dts}(F)),\ \circ \in \{;, [], \|\}$;
3. $Box_{dts}(E[f]) = \Theta_{[f]}(Box_{dts}(E))$;
4. $Box_{dts}(E \circ a) = \Theta_{\circ a}(Box_{dts}(E)),\ \circ \in \{\text{rs}, \text{sy}\}$;
5. $Box_{dts}([E * F * K]) = \Theta_{[**]}(Box_{dts}(E), Box_{dts}(F), Box_{dts}(K))$.

For $E \in RegStatExpr$, let $Box_{dts}(\overline{E}) = \overline{Box_{dts}(E)}$ and $Box_{dts}(\underline{E}) = \underline{Box_{dts}(E)}$.

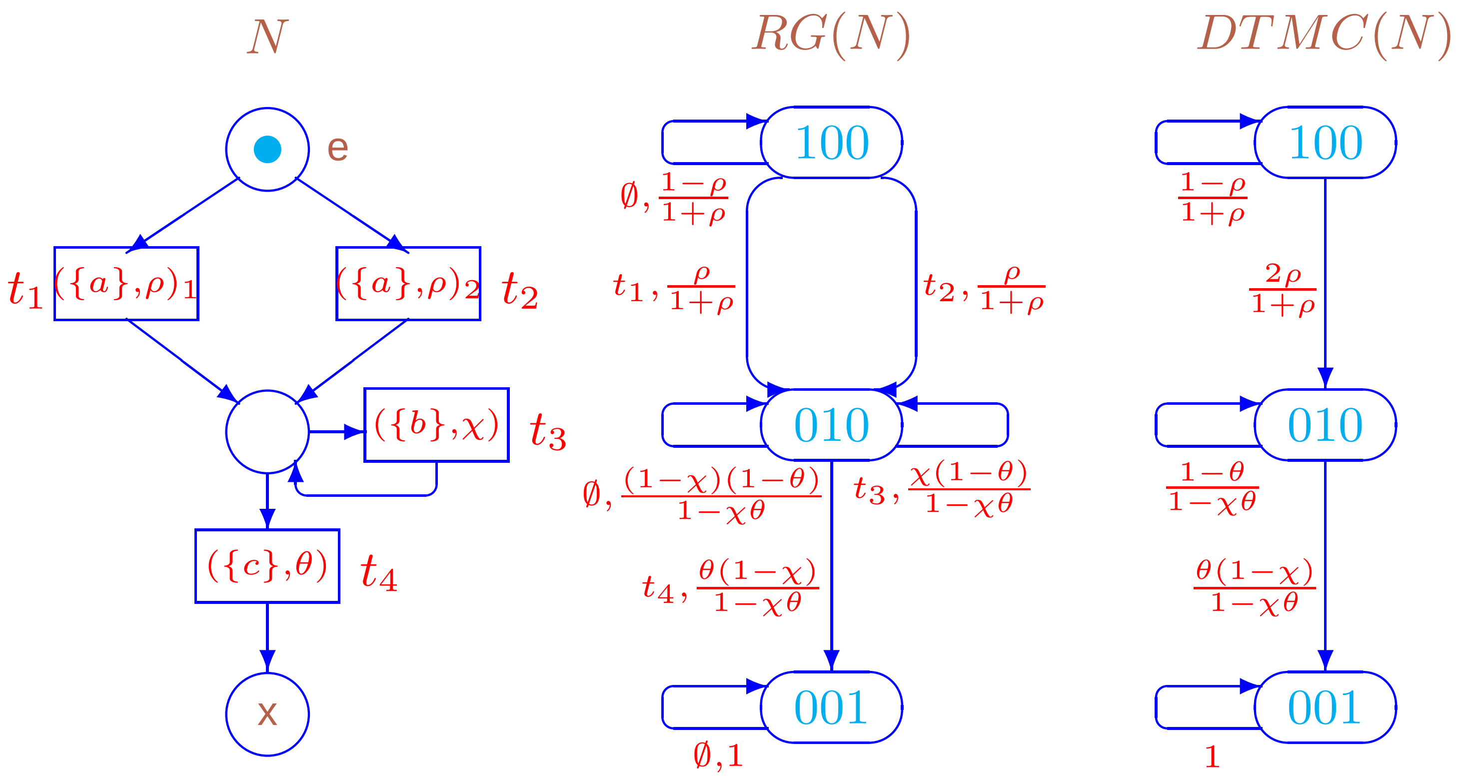
We denote isomorphism of transition systems by $\simeq$,

and the same symbol denotes isomorphism of reachability graphs and DTMCs

as well as isomorphism between transition systems and reachability graphs.

Theorem 1 *For any static expression* E *we have* $TS(\overline{E}) \simeq RG(Box_{dts}(\overline{E}))$.

Proposition 1 *For any static expression* E *we have* $DTMC(\overline{E}) \simeq DTMC(Box_{dts}(\overline{E}))$.



BOXIT: The marked dts-box $N = Box_{dts}(\overline{E})$ for $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$, its reachability graph and the underlying DTMC

Stochastic equivalences

Step stochastic bisimulation equivalence

We consider $\mathcal{L}(\Gamma) \in I\!N_f^{\mathcal{L}}$ for $\Gamma \in I\!N_f^{\mathcal{SL}}$, i.e., the multisets of multiactions.

Let G be a dynamic expression and $\mathcal{H} \subseteq DR(G)$. For $s \in DR(G)$ and $A \in I\!N_f^{\mathcal{L}}$ we write $s \xrightarrow{A}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = PM_A(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via steps with the multiaction part A*:

$$PM_A(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \xrightarrow{\Gamma} \tilde{s},\ \mathcal{L}(\Gamma) = A\}} PT(\Gamma, s).$$

We write $s \xrightarrow{A} \mathcal{H}$ if $\exists \mathcal{P}\ s \xrightarrow{A}_{\mathcal{P}} \mathcal{H}$.

We write $s \rightarrow_{\mathcal{P}} \mathcal{H}$ if $\exists A\ s \xrightarrow{A} \mathcal{H}$, where $\mathcal{P} = PM(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via any steps*:

$$PM(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \xrightarrow{\Gamma} \tilde{s}\}} PT(\Gamma, s).$$

Definition 13 *Let G and G' be dynamic expressions. An equivalence relation $\mathcal{R} \subseteq (DR(G) \cup DR(G'))^2$ is a step stochastic bisimulation between G and G', $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$, if:*

1. $([G]_\approx, [G']_\approx) \in \mathcal{R}$.
2. $(s_1, s_2) \in \mathcal{R} \;\Rightarrow\; \forall \mathcal{H} \in (DR(G) \cup DR(G'))/_\mathcal{R} \; \forall A \in I\!N_f^\mathcal{L}$

$$s_1 \stackrel{A}{\rightarrow}_\mathcal{P} \mathcal{H} \;\Leftrightarrow\; s_2 \stackrel{A}{\rightarrow}_\mathcal{P} \mathcal{H}.$$

Two dynamic expressions G and G' are step stochastic bisimulation equivalent, $G \underline{\leftrightarrow}_{ss} G'$, if $\exists \mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$.

$\mathcal{R}_{ss}(G, G') = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{ss} G'\}$ is the *union of all step stochastic bisimulations* between G and G'.

Proposition 2 *Let G and G' be dynamic expressions and $G \underline{\leftrightarrow}_{ss} G'$. Then $\mathcal{R}_{ss}(G, G')$ is the largest step stochastic bisimulation between G and G'.*

Interrelations of the stochastic equivalences

$$\underline{\leftrightarrow}_{ss} \longleftarrow =_{ts} \longleftarrow \approx$$

Interrelations of the stochastic equivalences

Theorem 2 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\twoheadrightarrow \in \{\underline{\leftrightarrow}, =, \approx\}$ *and* $\star, \star\star \in \{\_, ss, ts\}$. *For dynamic expressions* G *and* G'

$$G \leftrightarrow_{\star} G' \ \Rightarrow\ G \twoheadleftarrow\!\!\twoheadrightarrow_{\star\star} G'$$

iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\twoheadrightarrow_{\star\star}$.

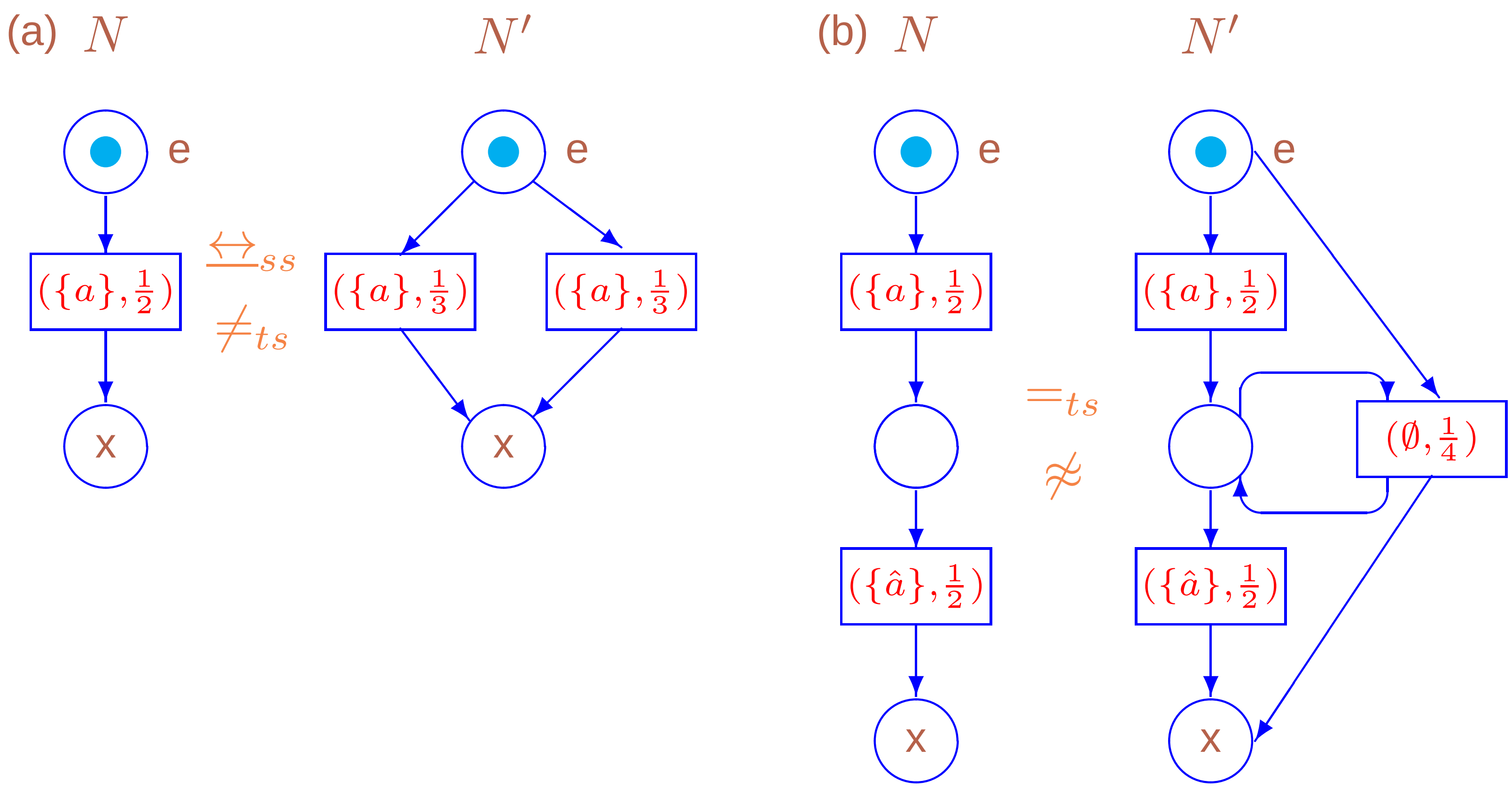
Validity of the implications

- The implication $=_{ts} \to \underline{\leftrightarrow}_{ss}$ is proved as follows. Let $\beta : G =_{ts} G'$. Then $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$, where $\mathcal{R} = \{(s, \beta(s)) \mid s \in DR(G)\}$.
- The implication $\approx \to =_{ts}$ is valid, since the transition system of a dynamic formula is defined based on its structural equivalence class.

Absence of the additional nontrivial arrows

(a) Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{3})_1[](\{a\}, \frac{1}{3})_2$. Then $\overline{E} \underline{\leftrightarrow}_{ss} \overline{E'}$, but $\overline{E} \neq_{ts} \overline{E'}$, since $TS(\overline{E})$ has only one transition from the initial to the final state while $TS(\overline{E'})$ has two such ones.

(b) Let $E = (\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2}))$ sy a. Then $\overline{E} =_{ts} \overline{E'}$, but $\overline{E} \not\approx \overline{E'}$, since $\overline{E}$ and $\overline{E'}$ cannot be reached from each other by applying inaction rules.



Dts-boxes of the dynamic expressions from equivalence examples of the theorem above

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$ for each picture (a)–(b).

Reduction modulo equivalences

An *autobisimulation* is a bisimulation between an expression and itself.

For a dynamic expression G and a step stochastic autobisimulation $\mathcal{R} : G\underline{\leftrightarrow}_{ss}G$, let $\mathcal{K} \in DR(G)/_{\mathcal{R}}$ and $s_1, s_2 \in \mathcal{K}$.

We have $\forall\widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}}\ \forall A \in I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}\ s_1 \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}} \Leftrightarrow s_2 \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

The equality is valid for all $s_1, s_2 \in \mathcal{K}$, hence, we can rewrite it as $\mathcal{K}\stackrel{A}{\rightarrow}_{\mathcal{P}}\widetilde{\mathcal{K}}$, where $\mathcal{P} = PM_A(\mathcal{K}, \widetilde{\mathcal{K}}) = PM_A(s_1, \widetilde{\mathcal{K}}) = PM_A(s_2, \widetilde{\mathcal{K}})$.

We write $\mathcal{K}\stackrel{A}{\rightarrow}\widetilde{\mathcal{K}}$ if $\exists\mathcal{P}\ \mathcal{K} \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$ and $\mathcal{K}\rightarrow\widetilde{\mathcal{K}}$ if $\exists A\ \mathcal{K} \stackrel{A}{\rightarrow} \widetilde{\mathcal{K}}$.

The similar arguments: we write $\mathcal{K}\rightarrow_{\mathcal{P}}\widetilde{\mathcal{K}}$, where $\mathcal{P} = PM(\mathcal{K}, \widetilde{\mathcal{K}}) = PM(s_1, \widetilde{\mathcal{K}}) = PM(s_2, \widetilde{\mathcal{K}})$.

$\mathcal{R}_{ss}(G) = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{ss} G\}$ is the *largest step stochastic autobisimulation* on G.

Definition **14** *The* quotient (by $\underline{\leftrightarrow}_{ss}$) (labeled probabilistic) transition system *of a dynamic expression* G is $TS_{\underline{\leftrightarrow}_{ss}}(G) = (S_{\underline{\leftrightarrow}_{ss}}, L_{\underline{\leftrightarrow}_{ss}}, \mathcal{T}_{\underline{\leftrightarrow}_{ss}}, s_{\underline{\leftrightarrow}_{ss}})$*, where*

- $S_{\underline{\leftrightarrow}_{ss}} = DR(G)/_{\mathcal{R}_{ss}(G)}$;
- $L_{\underline{\leftrightarrow}_{ss}} \subseteq (I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}) \times (0;1]$;
- $\mathcal{T}_{\underline{\leftrightarrow}_{ss}} = \{(\mathcal{K}, (A, PM_A(\mathcal{K}, \widetilde{\mathcal{K}})), \widetilde{\mathcal{K}}) \mid \mathcal{K} \in DR(G)/_{\mathcal{R}_{ss}(G)},\ \mathcal{K} \stackrel{A}{\rightarrow} \widetilde{\mathcal{K}}\}$;
- $s_{\underline{\leftrightarrow}_{ss}} = \{[G]_{\approx}\}$.

The transition $(\mathcal{K}, (A, \mathcal{P}), \widetilde{\mathcal{K}}) \in \mathcal{T}_{\underline{\leftrightarrow}_{ss}}$ will be written as $\mathcal{K} \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

For $E \in RegStatExpr$, let $TS_{\underline{\leftrightarrow}_{ss}}(E) = TS_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Definition **15** *The* quotient (by $\underline{\leftrightarrow}_{ss}$) underlying DTMC *of a dynamic expression* G, $DTMC_{\underline{\leftrightarrow}_{ss}}(G)$, *has the state space* $DR(G)/_{\mathcal{R}_{ss}(G)}$ *and the transitions* $\mathcal{K} \rightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, *where* $\mathcal{P} = PM(\mathcal{K}, \widetilde{\mathcal{K}})$.

For $E \in RegStatExpr$, let $DTMC_{\underline{\leftrightarrow}_{ss}}(E) = DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Stationary behaviour

Theoretical background

The elements $\mathcal{P}_{ij}$ $(1 \leq i, j \leq n = |DR(G)|)$ of *(one-step) transition probability matrix (TPM)* $\mathbf{P}$ for $DTMC(G)$:

$$\mathcal{P}_{ij} = \begin{cases} PM(s_i, s_j), & s_i \rightarrow s_j; \\ 0, & \text{otherwise}. \end{cases}$$

The *transient (k-step, $k \in \mathbb{N}$) probability mass function (PMF)* $\psi[k] = (\psi_1[k], \ldots, \psi_n[k])$ for $DTMC(G)$ is the solution of $\psi[k] = \psi[0]\mathbf{P}^k$,

where $\psi[0] = (\psi_1[0], \ldots, \psi_n[0])$ is the *initial PMF*: $\psi_i[0] = \begin{cases} 1, & s_i = [G]_\approx; \\ 0, & \text{otherwise}. \end{cases}$

We have $\psi[k+1] = \psi[k]\mathbf{P},\ k \in \mathbb{N}$.

The *steady-state PMF* $\psi = (\psi_1, \ldots, \psi_n)$ for $DTMC(G)$ is the solution of $\begin{cases} \psi(\mathbf{P} - \mathbf{E}) = \mathbf{0} \\ \psi \mathbf{1}^T = 1 \end{cases}$,

where $\mathbf{0}$ is a vector with n values 0, $\mathbf{1}$ is that with n values 1.

When $DTMC(G)$ has the single steady state, $\psi = \lim_{k\to\infty} \psi[k]$.

For $s \in DR(G)$ with $s = s_i$ $(1 \le i \le n)$ we define $\psi[k](s) = \psi_i[k]$ $(k \in I\!N)$ and $\psi(s) = \psi_i$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G),\ S, \widetilde{S} \subseteq DR(G)$.

The following performance indices (measures) are based on the steady-state PMF.

- The *average recurrence (return) time in the state* s (the number of discrete time units or steps required for this) is $\frac{1}{\psi(s)}$.
- The *fraction of residence time in the state* s is $\psi(s)$.
- The *fraction of residence time in the set of states* $S \subseteq DR(G)$ or the *probability of the event determined by a condition that is true for all states from* S is $\sum_{s\in S} \psi(s)$.
- The *relative fraction of residence time in the set of states* S *w.r.t. that in* $\widetilde{S}$ is $\frac{\sum_{s\in S}\psi(s)}{\sum_{\tilde{s}\in\widetilde{S}}\psi(\tilde{s})}$.
- The *steady-state probability to perform a step with an activity* (α, ρ) is $\sum_{s\in DR(G)} \psi(s) \sum_{\{\Gamma|(\alpha,\rho)\in\Gamma\}} PT(\Gamma, s)$.
- The *probability of the event determined by a reward function* r *on the states* is $\sum_{s\in DR(G)} \psi(s) r(s)$.

Steady state and equivalences

Proposition 3 *Let G, G' be dynamic expressions with $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$. Then* $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

$$\sum_{s \in \mathcal{H} \cap DR(G)} \psi(s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \psi'(s').$$

Let G be a dynamic expression and ψ be the steady-state PMF for $DTMC(G)$, $\psi_{\underline{\leftrightarrow}_{ss}}$ be the steady-state PMF for $DTMC_{\underline{\leftrightarrow}_{ss}}(G)$.
By the proposition above: $\forall \mathcal{H} \in DR(G)/_{\mathcal{R}_{ss}(G)}$ $\psi_{\underline{\leftrightarrow}_{ss}}(\mathcal{H}) = \sum_{s \in \mathcal{H}} \psi(s)$.

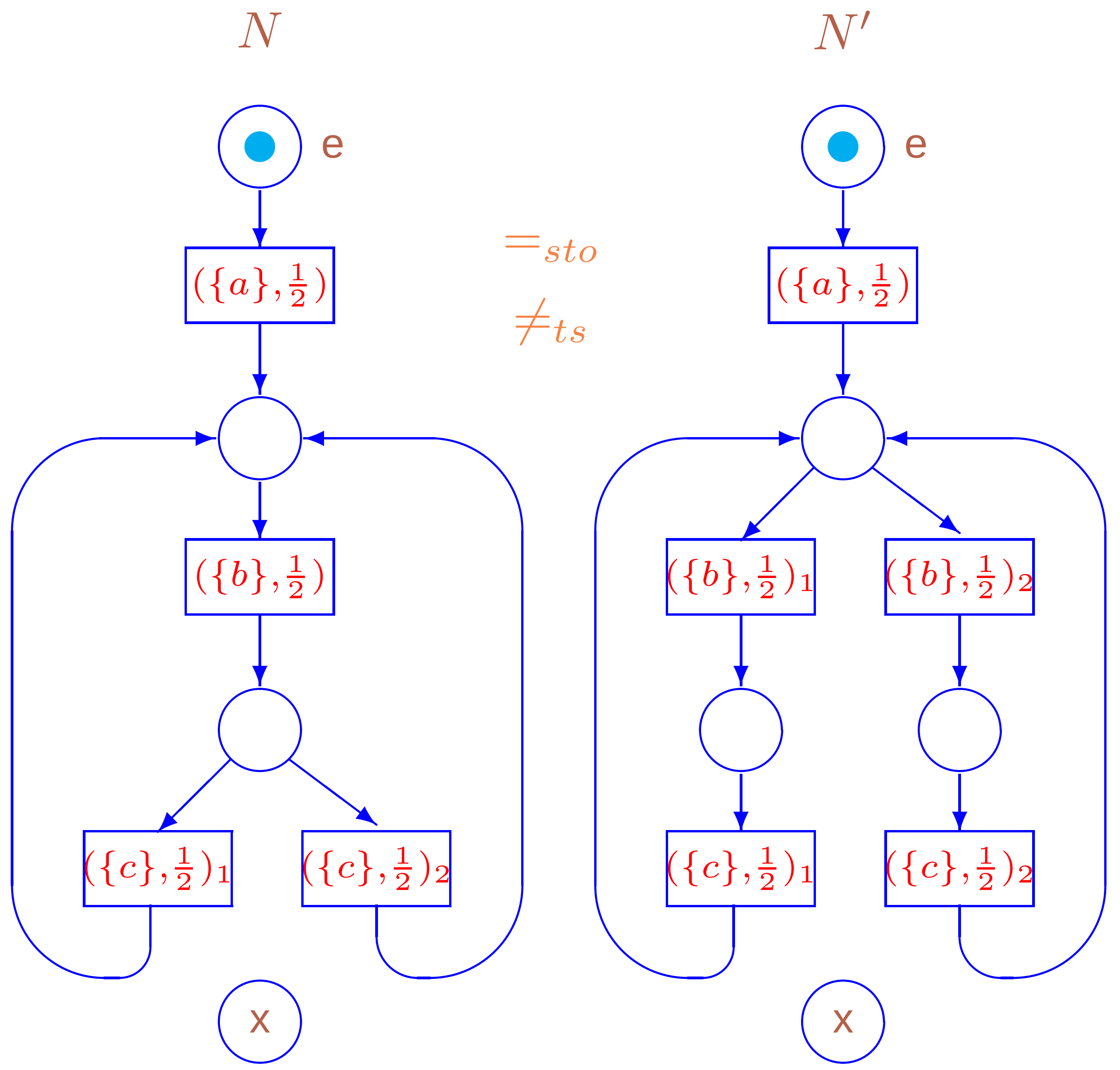
Definition 16 *A* derived step trace *of a dynamic expression* G *is* $\Sigma = A_1 \cdots A_n \in (\mathbb{N}_f^{\mathcal{L}})^*$, *where* $\exists s \in DR(G)\ s \stackrel{\Gamma_1}{\to} s_1 \stackrel{\Gamma_2}{\to} \cdots \stackrel{\Gamma_n}{\to} s_n,\ \mathcal{L}(\Gamma_i) = A_i\ (1 \le i \le n)$.

The probability to execute the derived step trace Σ in s:

$$PT(\Sigma, s) = \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid s=s_0 \stackrel{\Gamma_1}{\to} s_1 \stackrel{\Gamma_2}{\to} \cdots \stackrel{\Gamma_n}{\to} s_n,\ \mathcal{L}(\Gamma_i)=A_i\ (1\le i\le n)\}} \prod_{i=1}^{n} PT(\Gamma_i, s_{i-1}).$$

Theorem 3 *Let* G, G' *be dynamic expressions with* $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ *and* Σ *be a derived step trace of* G *and* G'. *Then* $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

$$\sum_{s\in\mathcal{H}\cap DR(G)} \psi(s) PT(\Sigma, s) = \sum_{s'\in\mathcal{H}\cap DR(G')} \psi'(s') PT(\Sigma, s').$$



$\underline{\leftrightarrow}_{ss}$ implies a coincidence of step trace probabilities

$\mathsf{Stop} = (\{c\}, \frac{1}{2})$ rs c is the process that performs empty loops with probability 1 and never terminates.

Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2})_1[](\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]$ and
$E' = [(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]$.
We have $\overline{E} =_{sto} \overline{E'}$, hence, $\overline{E} \underline{\leftrightarrow}_{ss} \overline{E'}$.

$DR(\overline{E})$ consists of

$s_1 = [[\overline{(\{a\}, \frac{1}{2})} * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2})_1[](\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s_2 = [[(\{a\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{1}{2})}; ((\{c\}, \frac{1}{2})_1[](\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); \overline{((\{c\}, \frac{1}{2})_1[](\{c\}, \frac{1}{2})_2)}) * \mathsf{Stop}]]_\approx$.

$DR(\overline{E'})$ consists of

$s'_1 = [[\overline{(\{a\}, \frac{1}{2})} * (((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s'_2 = [[(\{a\}, \frac{1}{2}) * \overline{(((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))} * \mathsf{Stop}]]_\approx$,
$s'_3 = [[(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{2})_1; \overline{(\{c\}, \frac{1}{2})_1})[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s'_4 = [[(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; \overline{(\{c\}, \frac{1}{2})_2})) * \mathsf{Stop}]]_\approx$.

The steady-state PMFs ψ for $DTMC(\overline{E})$ and ψ' for $DTMC(\overline{E'})$ are

$$\psi = \left(0, \frac{1}{2}, \frac{1}{2}\right),\ \psi' = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Consider $\mathcal{H} = \{s_3, s_3', s_4'\}$. The steady-state probabilities for $\mathcal{H}$ coincide:
$\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi(s) = \psi(s_3) = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \psi'(s_3') + \psi'(s_4') = \sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} \psi'(s')$.

Let $\Sigma = \{\{c\}\}$. The steady-state probabilities to come in the equivalence class $\mathcal{H}$ and start the step trace Σ from it coincide as well: $\psi(s_3)(PT(\{(\{c\}, \frac{1}{2})_1\}, s_3) + PT(\{(\{c\}, \frac{1}{2})_2\}, s_3)) =$
$\frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2} = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 = \psi'(s_3')PT(\{(\{c\}, \frac{1}{2})_1\}, s_3') + \psi'(s_4')PT(\{(\{c\}, \frac{1}{2})_2\}, s_4')$.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.

Simplification of performance analysis

The method of performance analysis simplification.

1. The system under investigation is specified by a static expression of $dtsPBC$.

2. The transition system of the expression is constructed.

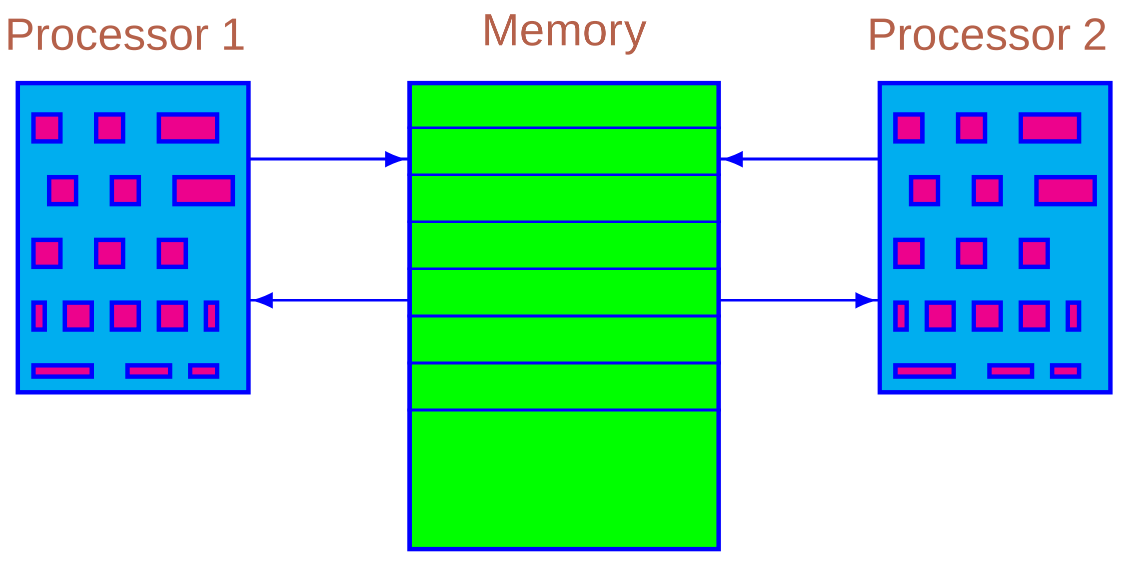
3. After examining this transition system for self-similarity and symmetry, a step stochastic autobisimulation equivalence for the expression is determined.

4. The quotient underlying DTMC of the expression is constructed.

5. The steady-state probabilities and performance indices based on this DTMC are calculated.

Shared memory system

A model of two processors accessing a common shared memory [MBCDF95]



The diagram of the shared memory system

After activation of the system, two processors are active, and the common memory is available. Each processor can request an access to the memory.

When a processor starts an acquisition of the memory, another processor waits until the former one ends its operations, and the system returns to the state with both active processors and the available memory.

a corresponds to the system activation.

r_i $(1 \leq i \leq 2)$ represent the common memory request of processor i.

b_i and e_i correspond to the beginning and the end of the common memory access of processor i.

The other actions are used for communication purpose only.

The static expression of the first processor is

$E_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r_1\}, \frac{1}{2}); (\{b_1, y_1\}, \frac{1}{2}); (\{e_1, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$E_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r_2\}, \frac{1}{2}); (\{b_2, y_2\}, \frac{1}{2}); (\{e_2, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is

$E_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, \frac{1}{2}); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \frac{1}{2}); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the shared memory system with two processors is

$E = (E_1 \| E_2 \| E_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

Interpretation of the states

s_1: the initial state,

s_2: the system is activated and the memory is not requested,

s_3: the memory is requested by the first processor,

s_4: the memory is requested by the second processor,

s_5: the memory is allocated to the first processor,

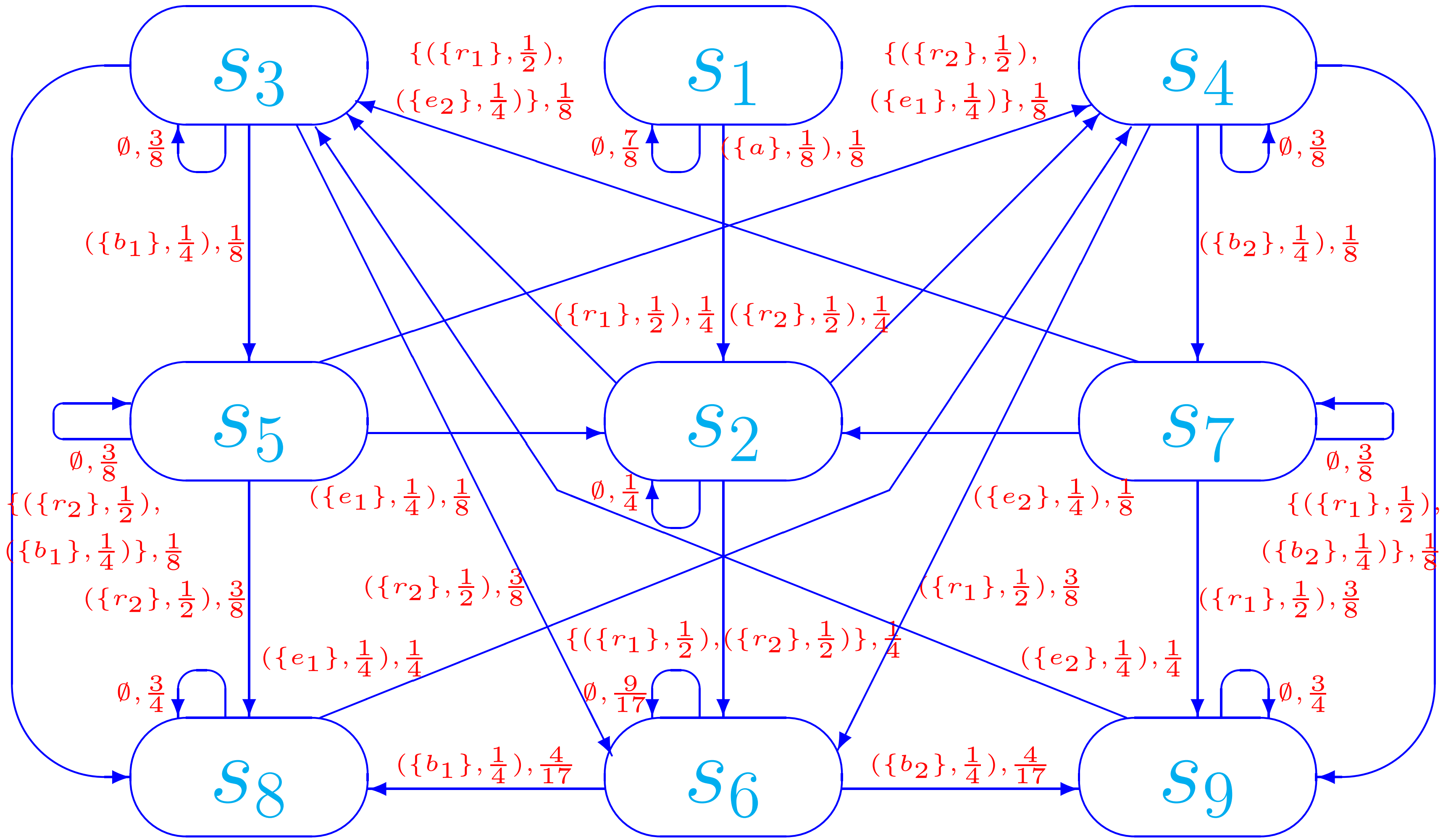
s_6: the memory is requested by two processors,

s_7: the memory is allocated to the second processor,

s_8: the memory is allocated to the first processor and the memory is requested by the second processor,

s_9: the memory is allocated to the second processor and the memory is requested by the first processor.

$TS(\overline{E})$



The transition system of the shared memory system

The TPM for $DTMC(\overline{E})$ is

$$\mathbf{P} = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & 0 & \frac{1}{8} & \frac{3}{8} & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{3}{3} & 0 & \frac{3}{8} & \frac{1}{8} & 0 & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{3}{8} & 0 & 0 & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{17} & 0 & \frac{4}{17} & \frac{4}{17} \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{bmatrix}.$$

The steady-state PMF for $DTMC(\overline{E})$ is

$$\psi = \left(0, \frac{16}{2103}, \frac{80}{701}, \frac{80}{701}, \frac{16}{701}, \frac{391}{2103}, \frac{16}{701}, \frac{560}{2103}, \frac{560}{2103}\right).$$

The average sojourn time vector of $\overline{E}$ is

$$SJ = \left(8, \frac{4}{3}, \frac{8}{5}, \frac{8}{5}, \frac{8}{5}, \frac{17}{8}, \frac{8}{5}, 4, 4\right).$$

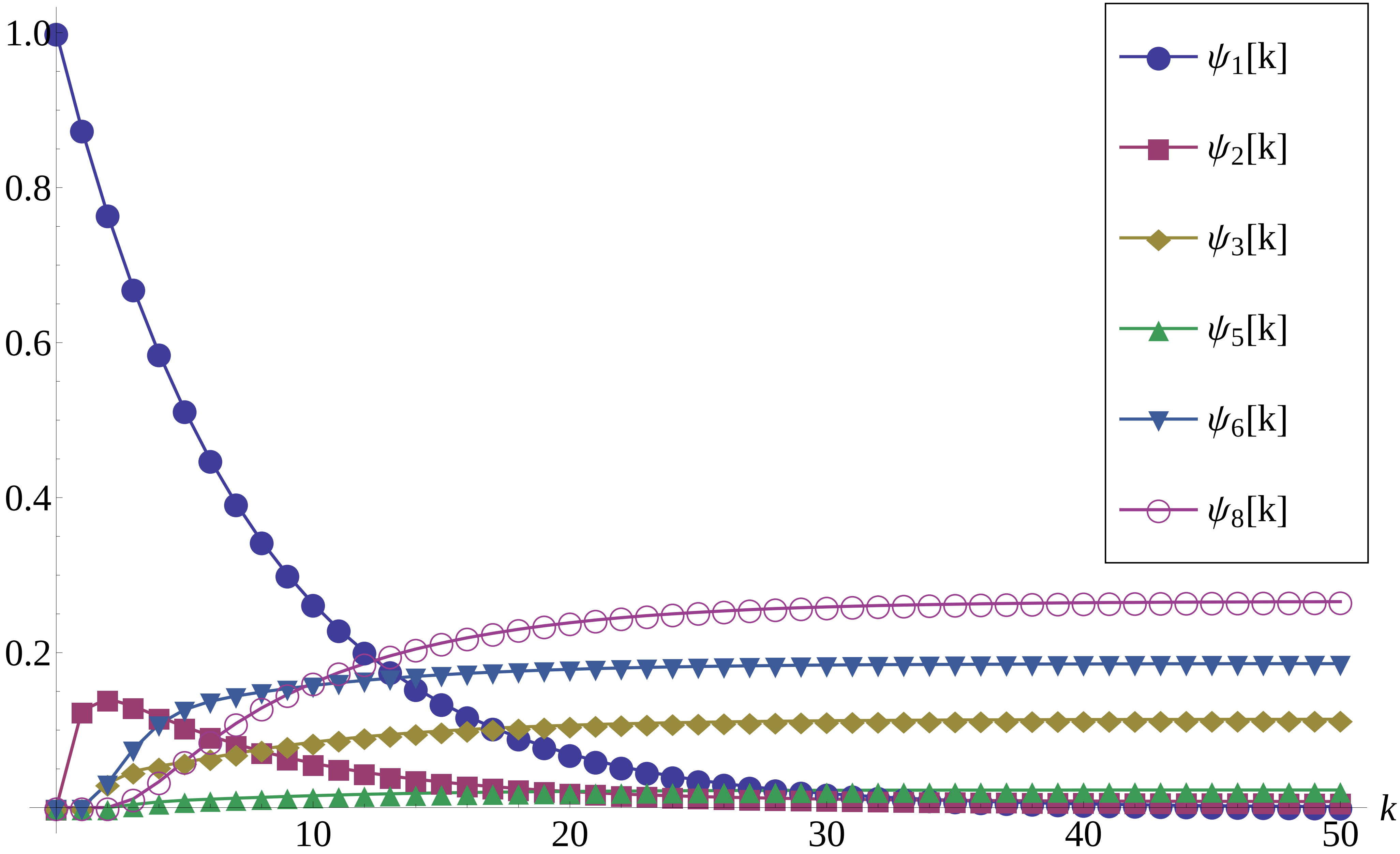
The sojourn time variance vector of $\overline{E}$ is

$$VAR = \left(56, \frac{4}{9}, \frac{24}{25}, \frac{24}{25}, \frac{24}{25}, \frac{153}{64}, \frac{24}{25}, 12, 12\right).$$

Transient and steady-state probabilities of the shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	∞
$\psi_1[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	0
$\psi_2[k]$	0	0.1045	0.0573	0.0331	0.0207	0.0143	0.0110	0.0094	0.0085	0.0081	0.0078	0.0076
$\psi_3[k]$	0	0.0587	0.0845	0.0989	0.1063	0.1101	0.1121	0.1131	0.1136	0.1138	0.1140	0.1141
$\psi_5[k]$	0	0.0094	0.0154	0.0190	0.0209	0.0218	0.0223	0.0226	0.0227	0.0228	0.0228	0.0228
$\psi_6[k]$	0	0.1265	0.1577	0.1714	0.1785	0.1821	0.1840	0.1849	0.1854	0.1857	0.1858	0.1859
$\psi_8[k]$	0	0.0599	0.1611	0.2123	0.2386	0.2521	0.2590	0.2626	0.2644	0.2653	0.2658	0.2663

We depict the probabilities for the states $s_1, s_2, s_3, s_5, s_6, s_8$ only, since the corresponding values coincide for s_3, s_4 as well as for s_5, s_7 and for s_8, s_9.



Transient probabilities alteration diagram of the shared memory system

Performance indices

- The average recurrence time in the state s_2, the *average system run-through*, is $\frac{1}{\psi_2} = \frac{2103}{16} = 131\frac{7}{16}$.

- The common memory is available in the states s_2, s_3, s_4, s_6 only.

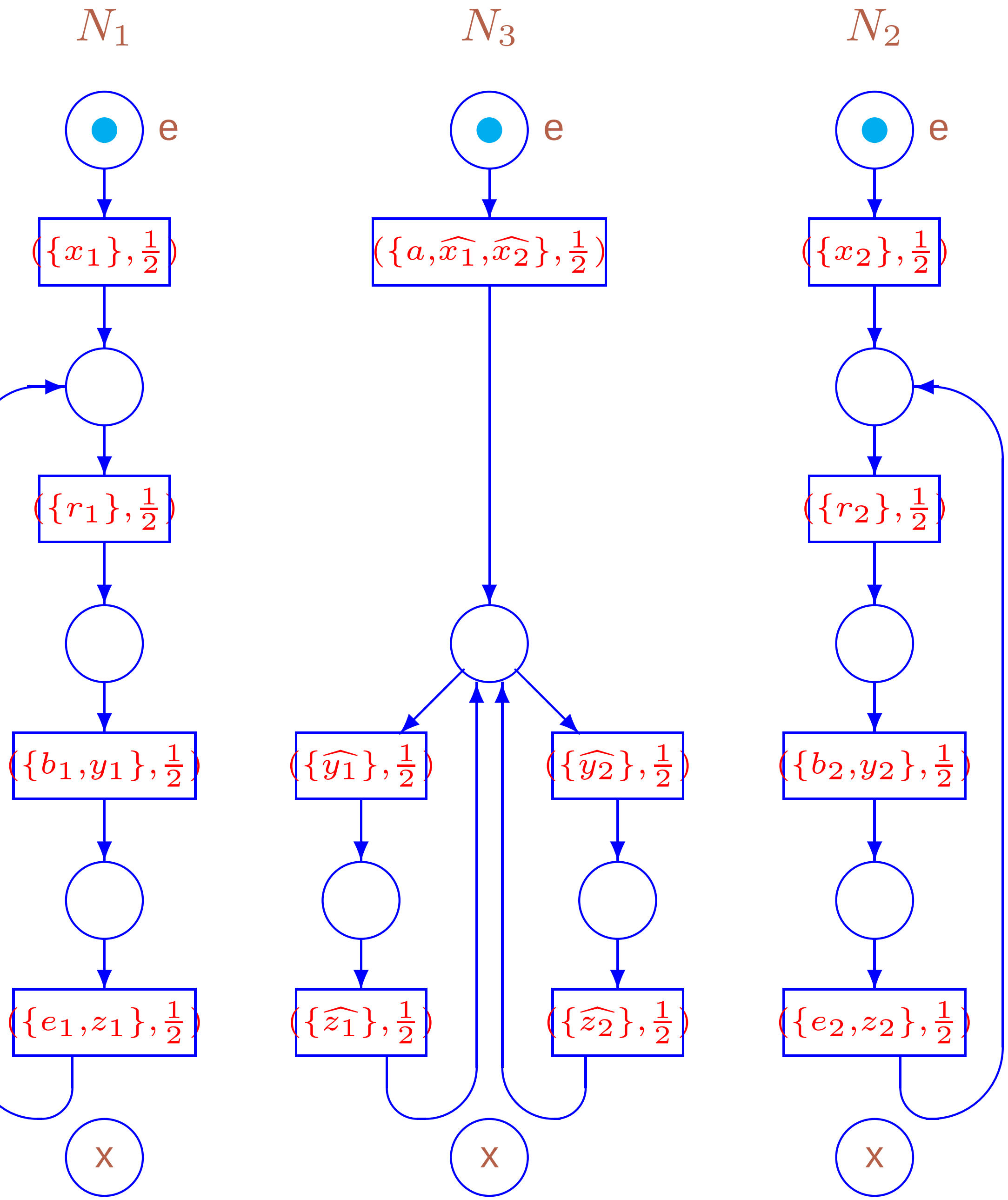
  The steady-state probability that the memory is available is $\psi_2 + \psi_3 + \psi_4 + \psi_6 = \frac{887}{2103}$.

  The steady-state probability that the memory is used, the *shared memory utilization*, is $1 - \frac{887}{2103} = \frac{1216}{2103}$.

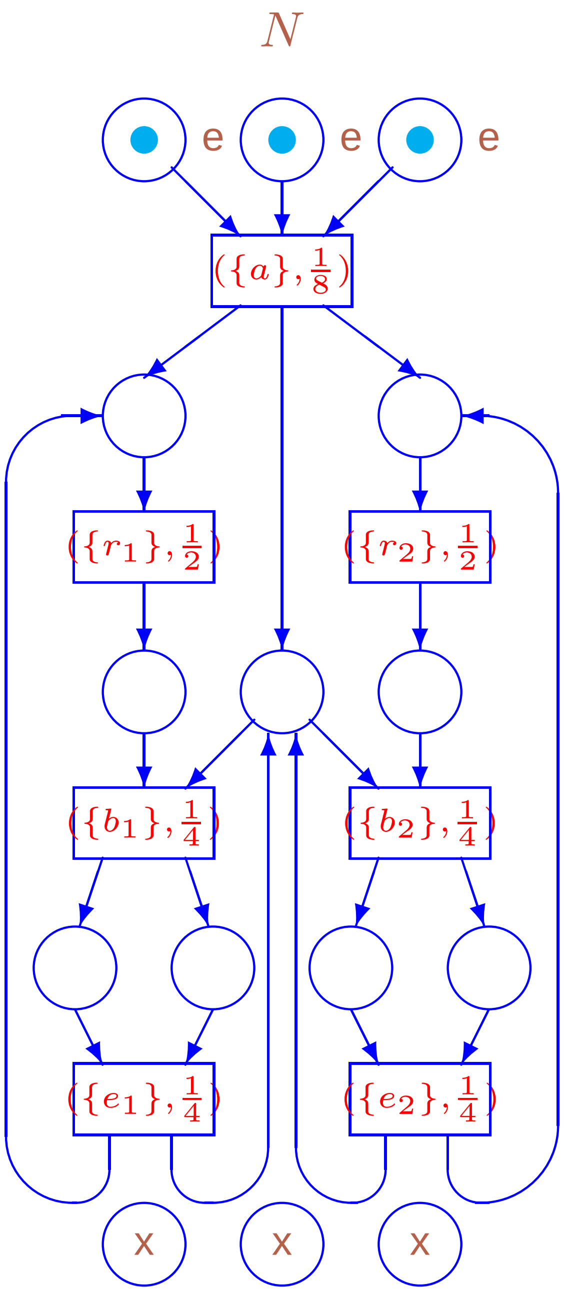
- The common memory request of the first processor $(\{r_1\}, \frac{1}{2})$ is only possible from the states s_2, s_4, s_7.

  The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from the first processor* is
  $\psi_2 \sum_{\{\Gamma | (\{r_1\}, \frac{1}{2}) \in \Gamma\}} PT(\Gamma, s_2) + \psi_4 \sum_{\{\Gamma | (\{r_1\}, \frac{1}{2}) \in \Gamma\}} PT(\Gamma, s_4) +$
  $\psi_7 \sum_{\{\Gamma | (\{r_1\}, \frac{1}{2}) \in \Gamma\}} PT(\Gamma, s_7) =$
  $\frac{16}{2103}\left(\frac{1}{4} + \frac{1}{4}\right) + \frac{80}{701}\left(\frac{3}{8} + \frac{1}{8}\right) + \frac{16}{701}\left(\frac{3}{8} + \frac{1}{8}\right) = \frac{152}{2103}$.



The marked dts-boxes of two processors and shared memory



The marked dts-box of the shared memory system

The abstract system

The static expression of the first processor is

$F_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{b, y_1\}, \frac{1}{2}); (\{e, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$F_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{b, y_2\}, \frac{1}{2}); (\{e, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is

$F_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, \frac{1}{2}); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \frac{1}{2}); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the abstract shared memory system with two processors is

$F = (F_1 \| F_2 \| F_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS(\overline{F})$ is similar to $TS(\overline{E})$.

$DTMC(\overline{F}) = DTMC(\overline{E})$, thus, the TPM and the steady-state PMF for $DTMC(\overline{F})$ and $DTMC(\overline{E})$ coincide.

Performance indices

The first and second performance indices are the same for the standard and abstract systems.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\}, \frac{1}{2})$ is only possible from the states s_2, s_3, s_4, s_5, s_7.

  The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from a processor* is

  $\psi_2 \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT(\Gamma, s_2) + \psi_3 \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT(\Gamma, s_3) +$
  $\psi_4 \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT(\Gamma, s_4) + \psi_5 \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT(\Gamma, s_5) +$
  $\psi_7 \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT(\Gamma, s_7) =$
  $\frac{16}{2103}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) + \frac{80}{701}\left(\frac{3}{8}+\frac{1}{8}\right) + \frac{80}{701}\left(\frac{3}{8}+\frac{1}{8}\right) + \frac{16}{701}\left(\frac{3}{8}+\frac{1}{8}\right) + \frac{16}{701}\left(\frac{3}{8}+\frac{1}{8}\right) = \frac{100}{701}$.

The quotient of the abstract system

$DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6\}$, where

$\mathcal{K}_1 = \{s_1\}$ (the initial state),

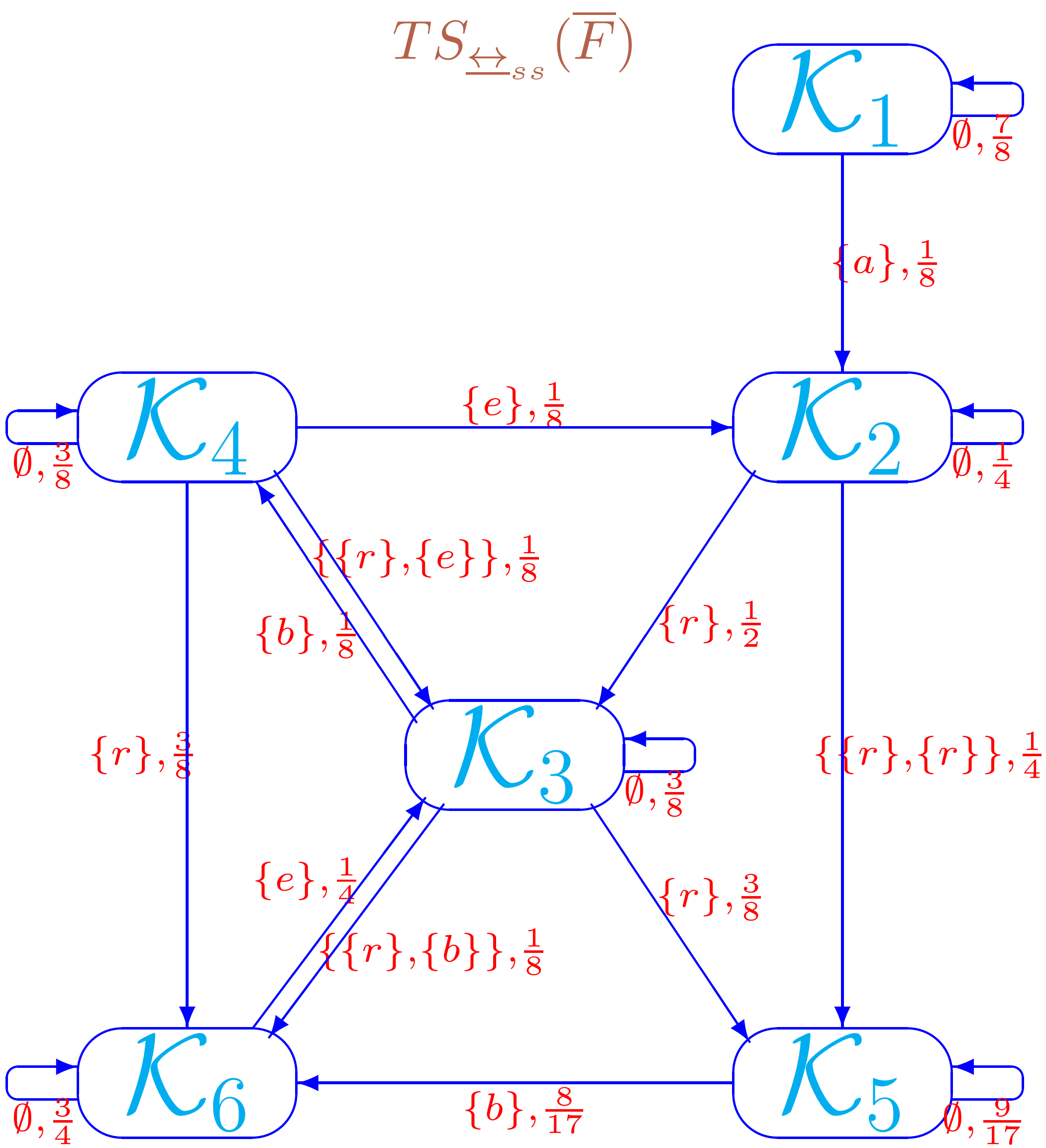
$\mathcal{K}_2 = \{s_2\}$ (the system is activated and the memory is not requested),

$\mathcal{K}_3 = \{s_3, s_4\}$ (the memory is requested by one processor),

$\mathcal{K}_4 = \{s_5, s_7\}$ (the memory is allocated to a processor),

$\mathcal{K}_5 = \{s_6\}$ (the memory is requested by two processors),

$\mathcal{K}_6 = \{s_8, s_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).



The quotient transition system of the abstract shared memory system

The TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

$$\mathbf{P}' = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{3}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & \frac{9}{17} & \frac{8}{17} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{bmatrix}.$$

The steady-state PMF for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

$$\psi' = \left(0, \frac{16}{2103}, \frac{160}{701}, \frac{32}{701}, \frac{391}{2103}, \frac{1120}{2103}\right).$$

The quotient average sojourn time vector of $\overline{F}$ is

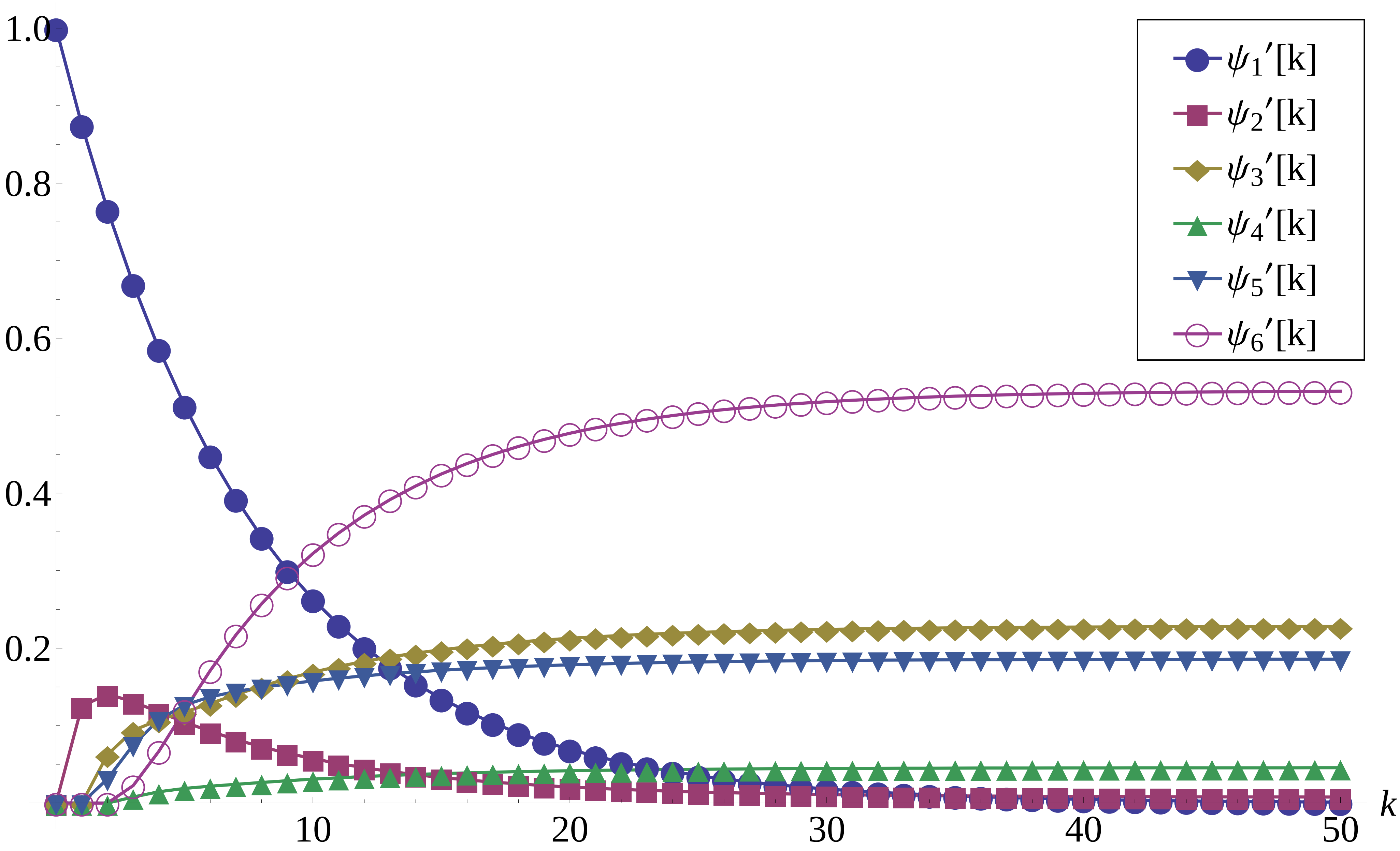
$$SJ' = \left(8, \frac{4}{3}, \frac{8}{5}, \frac{8}{5}, \frac{17}{8}, 4\right).$$

The quotient sojourn time variance vector of $\overline{F}$ is

$$VAR' = \left(56, \frac{4}{9}, \frac{24}{25}, \frac{24}{25}, \frac{153}{64}, 12\right).$$

Transient and steady-state probabilities of the quotient abstract shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	∞
$\psi'_1[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	0
$\psi'_2[k]$	0	0.1045	0.0573	0.0331	0.0207	0.0143	0.0110	0.0094	0.0085	0.0081	0.0078	0.0076
$\psi'_3[k]$	0	0.1175	0.1690	0.1979	0.2127	0.2203	0.2241	0.2261	0.2272	0.2277	0.2280	0.2282
$\psi'_4[k]$	0	0.0189	0.0309	0.0381	0.0418	0.0437	0.0446	0.0451	0.0454	0.0455	0.0456	0.0456
$\psi'_5[k]$	0	0.1265	0.1577	0.1714	0.1785	0.1821	0.1840	0.1849	0.1854	0.1857	0.1858	0.1859
$\psi'_6[k]$	0	0.1197	0.3221	0.4247	0.4772	0.5042	0.5180	0.5251	0.5287	0.5306	0.5316	0.5326



Transient probabilities alteration diagram of the quotient abstract shared memory system

Performance indices

- The average recurrence time in the state $\mathcal{K}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\psi'_2} = \frac{2103}{16} = 131\frac{7}{16}$.
- The common memory is available in the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_5$ only.

  The steady-state probability that the memory is available is $\psi'_2 + \psi'_3 + \psi'_5 = \frac{16}{2103} + \frac{160}{701} + \frac{391}{2103} = \frac{887}{2103}$.

  The steady-state probability that the memory is used (i.e., not available), the *shared memory utilization*, is $1 - \frac{887}{2103} = \frac{1216}{2103}$.
- The common memory request of a processor $\{r\}$ is only possible from the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$.

  The request probability in each of the states is a sum of execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is

  $$\psi'_2 \sum_{\{A,\widetilde{\mathcal{K}} \mid \{r\}\in A,\ \mathcal{K}_2 \stackrel{A}{\to} \widetilde{\mathcal{K}}\}} PM_A(\mathcal{K}_2, \widetilde{\mathcal{K}}) + \psi'_3 \sum_{\{A,\widetilde{\mathcal{K}} \mid \{r\}\in A,\ \mathcal{K}_3 \stackrel{A}{\to} \widetilde{\mathcal{K}}\}} PM_A(\mathcal{K}_3, \widetilde{\mathcal{K}}) + \psi'_4 \sum_{\{A,\widetilde{\mathcal{K}} \mid \{r\}\in A,\ \mathcal{K}_4 \stackrel{A}{\to} \widetilde{\mathcal{K}}\}} PM_A(\mathcal{K}_4, \widetilde{\mathcal{K}}) =$$

  $$\frac{16}{2103}\left(\frac{1}{2}+\frac{1}{4}\right) + \frac{160}{701}\left(\frac{3}{8}+\frac{1}{8}\right) + \frac{32}{701}\left(\frac{3}{8}+\frac{1}{8}\right) = \frac{100}{701}.$$

The performance indices are the same for the complete and the quotient abstract shared memory systems.

The coincidence of the first and second performance indices illustrates the result of proposition about steady-state probabilities.

The coincidence of the third performance index theorem about step traces from steady states:

one should apply its result to the step traces $\{\{r\}\},\ \{\{r\},\{r\}\},\ \{\{r\},\{b\}\},\ \{\{r\},\{e\}\}$ of $\overline{F}$ and itself,

and sum the left and right parts of the three resulting equalities.

Overview and open questions

The results obtained

- A discrete time stochastic extension $dtsPBC$ of finite PBC enriched with iteration.
- The step operational semantics based on labeled probabilistic transition systems.
- The denotational semantics in terms of a subclass of LDTSPNs.
- The method of performance evaluation based on underlying DTMCs.
- Step stochastic bisimulation equivalence of the expressions and dts-boxes.
- The transition systems and DTMCs reduction modulo the equivalence.
- A comparison of stationary behaviour up to the equivalence.
- Performance analysis simplification with the equivalence.
- The case study: the shared memory system.

Further research

- Introducing the deterministically timed multiactions with fixed time delays (including the zero delay).
- Extending the syntax with recursion operator.

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The slides can be downloaded from Internet:

http://itar.iis.nsk.su/files/itar/pages/dort11sld.pdf

Thank you for your attention!