Algebra $dtsPBC$: a discrete time stochastic extension of Petri box calculus

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Abstract: In [MVF01], a continuous time stochastic extension $sPBC$ of finite Petri box calculus PBC [BDH92] was proposed. In [MVCC03], iteration operator was added to $sPBC$.

Algebra $sPBC$ has an interleaving semantics, but PBC has a step one.

We constructed a discrete time stochastic extension $dtsPBC$ of finite PBC [Tar05] and enriched it with iteration [Tar06].

The step operational semantics is defined in terms of labeled probabilistic transition systems.

The denotational semantics is defined in terms of a subclass of labeled DTSPNs (LDTSPNs) called discrete time stochastic Petri boxes (dts-boxes).

We propose a variety of stochastic equivalences and investigate their interrelations.

It is explained how to use the equivalences for transition systems and discrete time Markov chains reduction.

A logical characterization of the equivalences is presented via probabilistic modal logics.

We demonstrate how to apply the equivalences to compare stationary behaviour.

A congruence relation is defined. The case studies of performance evaluation are presented.

Keywords: stochastic Petri net, stochastic process algebra, Petri box calculus, iteration, discrete time, transition systems, operational semantics, dts-box, denotational semantics, empty loop, stochastic equivalence, reduction, modal logic, stationary behaviour, congruence, performance evaluation.

Contents

Introduction

Previous work

- Continuous time (subsets of $\mathbb{R}_{\geq 0}$): interleaving semantics
  - *Continuous time stochastic Petri nets (CTSPNs)* [Mol82,FN85]:

    exponential transition firing delays,

    Continuous time Markov chain (CTMC).
  - *Generalized stochastic Petri nets (GSPNs)* [MCB84,CMBC93]:

    exponential and zero transition firing delays,

    Semi-Markov chain (SMC).

- Discrete time (subsets of $\mathbb{N}$): interleaving and step semantics
  - *Discrete time stochastic Petri nets (DTSPNs)* [Mol85,ZG94]:
    geometric transition firing delays,
    Discrete time Markov chain (DTMC).
  - *Discrete time deterministic and stochastic Petri nets (DTDSPNs)* [ZFH01]:
    geometric and fixed transition firing delays,
    Semi-Markov chain (SMC).
  - *Discrete deterministic and stochastic Petri nets (DDSPNs)* [ZCH97]:
    phase and fixed transition firing delays,
    Semi-Markov chain (SMC).

Stochastic process algebras

- $MTIPP$ [HR94]
- $GSPA$ [BKLL95]
- $PEPA$ [Hil96]
- $S\pi$ [Pri96]
- $EMPA$ [BGo98]
- $GSMPA$ [BBGo98]
- $sACP$ [AHR00]
- TCP^{dst} [MVi08]

More stochastic process calculi

- $TIPP$ [GHR93]
- $TPCCS$ [Han94]
- $PM-TIPP$ [Ret95]
- PPA [NFL95]
- $prBPA$, ACP_{π}^{+} [And99]
- $StAFP_0$ [BT01]
- $SM-PEPA$ [Brad05]
- $iPEPA$ [HBC13]

Algebra PBC and its extensions

- *Petri box calculus* PBC [BDH92]
- *Time Petri box calculus* $tPBC$ [Kou00]
- *Timed Petri box calculus* $TPBC$ [MF00]
- *Stochastic Petri box calculus* $sPBC$ [MVF01,MVCC03]
- *Ambient Petri box calculus* $APBC$ [FM03]
- *Arc time Petri box calculus* $atPBC$ [Nia05]
- *Generalized stochastic Petri box calculus* $gsPBC$ [MVCR08]
- *Discrete time stochastic Petri box calculus* $dtsPBC$ [Tar05,Tar06]
- *Discrete time stochastic and immediate Petri box calculus* $dtsiPBC$ [TMV10,TMV13]

Classification of stochastic process algebras

Time	Interleaving semantics	Non-interleaving semantics
Continuous	$MTIPP$ (CTMC), $PEPA$ (CTMP), $EMPA$ (SMC, CTMC), $sPBC$ (CTMC), $gsPBC$ (SMC)	$GSPA$ (GSMP), $S\pi$, $GSMPA$ (GSMP)
Discrete	TCP^{dst} (DTMRC)	$sACP$, $dtsPBC$ (DTMC), $dtsiPBC$ (SMC, DTMC)

The SPNs-based denotational semantics: orange SPA names.

The underlying stochastic process: in parentheses near the SPA names.

Transition labeling

- CTSPNs [Buc95]
- GSPNs [Buc98]
- DTSPNs [BT00]

Stochastic equivalences

- Probabilistic transition systems (PTSs) [BM89,Chr90,LS91,BHe97,KN98]
- SPAs [HR94,Hil94,BGo98]
- Markov process algebras (MPAs) [Buc94,BKe01]
- CTSPNs [Buc95]
- GSPNs [Buc98]
- Stochastic automata (SAs) [Buc99]
- Stochastic event structures (SESs) [MCW03]

Syntax

The *set of all finite multisets* over X is $\mathbb{N}_{fin}^X$.

The *set of all subsets (powerset)* of X is 2^X.

$Act = \{a, b, \ldots\}$ is the set of *elementary actions*.

$\widehat{Act} = \{\hat{a}, \hat{b}, \ldots\}$ is the set of *conjugated actions (conjugates)* s.t. $a \neq \hat{a}$ and $\hat{\hat{a}} = a$.

$\mathcal{A} = Act \cup \widehat{Act}$ is the set of *all actions*.

$\mathcal{L} = \mathbb{N}_{fin}^{\mathcal{A}}$ is the set of *all multiactions*.

The *alphabet* of $\alpha \in \mathcal{L}$ is $\mathcal{A}(\alpha) = \{x \in \mathcal{A} \mid \alpha(x) > 0\}$.

An *activity (stochastic multiaction)* is a pair (α, ρ), where
$\alpha \in \mathcal{L}$ and $\rho \in (0; 1)$ is the *probability* of multiaction α.

$\mathcal{SL}$ is the set of *all activities*.

The *alphabet* of $(\alpha, \rho) \in \mathcal{SL}$ is $\mathcal{A}(\alpha, \rho) = \mathcal{A}(\alpha)$.

The *alphabet* of $\Gamma \in \mathbb{N}_{fin}^{\mathcal{SL}}$ is $\mathcal{A}(\Gamma) = \cup_{(\alpha,\rho)\in\Gamma}\mathcal{A}(\alpha)$.

For $(\alpha, \rho) \in \mathcal{SL}$, its *multiaction part* is $\mathcal{L}(\alpha, \rho) = \alpha$ and its *probability part* is $\Omega(\alpha, \rho) = \rho$.

The *multiaction part* of $\Gamma \in \mathbb{N}_{fin}^{\mathcal{SL}}$ is $\mathcal{L}(\Gamma) = \sum_{(\alpha,\rho)\in\Gamma} \alpha$.

The operations: *sequential execution* $;$, *choice* $[]$, *parallelism* $\|$, *relabeling* $[f]$, *restriction* rs, *synchronization* sy and *iteration* $[**]$.

Sequential execution and choice have the standard interpretation.

Parallelism does not include synchronization unlike that in standard process algebras.

Relabeling functions $f : \mathcal{A} \rightarrow \mathcal{A}$ are bijections preserving conjugates: $\forall x \in \mathcal{A}\ f(\hat{x}) = \widehat{f(x)}$.
For $\alpha \in \mathcal{L}$, let $f(\alpha) = \sum_{x \in \alpha} f(x)$. For $\Gamma \in I\!N_{fin}^{\mathcal{SL}}$, let $f(\Gamma) = \sum_{(\alpha,\rho) \in \Gamma} (f(\alpha), \rho)$.

Restriction over $a \in Act$: any process behaviour containing a or its conjugate $\hat{a}$ is not allowed.

Let $\alpha, \beta \in \mathcal{L}$ be two multiactions s.t. for $a \in Act$ we have $a \in \alpha$ and $\hat{a} \in \beta$, or $\hat{a} \in \alpha$ and $a \in \beta$.
Synchronization of α and β by a is $\alpha \oplus_a \beta = \gamma$:

$$\gamma(x) = \begin{cases} \alpha(x) + \beta(x) - 1, & x = a \text{ or } x = \hat{a}; \\ \alpha(x) + \beta(x), & \text{otherwise}. \end{cases}$$

In the iteration, the initialization subprocess is executed first,
then the body one is performed zero or more times, finally, the termination one is executed.

Static expressions specify the structure of processes.

Definition 1 *Let* $(\alpha, \rho) \in \mathcal{SL}$ *and* $a \in Act$. *A* static expression *of* $dtsPBC$ *is*

$$E ::= \ (\alpha, \rho) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}E{*}E].$$

$StatExpr$ is the set of *all static expressions* of $dtsPBC$.

Definition 2 *Let* $(\alpha, \rho) \in \mathcal{SL}$ *and* $a \in Act$. *A* regular static expression *of* $dtsPBC$ *is*

$$E ::= \ (\alpha, \rho) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}D{*}E],$$

where $D ::= \ (\alpha, \rho) \mid D;E \mid D[]D \mid D[f] \mid D \text{ rs } a \mid D \text{ sy } a \mid [D{*}D{*}E].$

$RegStatExpr$ is the set of *all regular static expressions* of $dtsPBC$.

Dynamic expressions specify the states of processes.

Dynamic expressions are obtained from static ones annotated with upper or lower bars.

The *underlying static expression* of a dynamic one: removing all upper and lower bars.

Definition 3 *Let $E \in StatExpr$ and $a \in Act$. A dynamic expression of $dtsPBC$ is*

$$G ::= \overline{E} \mid \underline{E} \mid G;E \mid E;G \mid G[]E \mid E[]G \mid G\|G \mid G[f] \mid G \text{ rs } a \mid G \text{ sy } a \mid [G*E*E] \mid [E*G*E] \mid [E*E*G].$$

$DynExpr$ is the set of *all dynamic expressions* of $dtsPBC$.

Definition 4 *A dynamic expression is regular if its underlying static expression is regular.*

$RegDynExpr$ is the set of *all regular dynamic expressions* of $dtsPBC$.

Operational semantics

Inaction rules

Inaction rules: instantaneous structural transformations.

Let $E, F, K \in RegStatExpr$ and $a \in Act$.

Inaction rules for overlined and underlined regular static expressions

$\overline{E;F} \Rightarrow \overline{E};F$	$\underline{E};F \Rightarrow E;\overline{F}$	$E;\underline{F} \Rightarrow \underline{E;F}$
$\overline{E[]F} \Rightarrow \overline{E}[]F$	$\overline{E[]F} \Rightarrow E[]\overline{F}$	$\underline{E}[]F \Rightarrow \underline{E[]F}$
$E[]\underline{F} \Rightarrow \underline{E[]F}$	$\overline{E\|F} \Rightarrow \overline{E}\|\overline{F}$	$\underline{E}\|\underline{F} \Rightarrow \underline{E\|F}$
$\overline{E[f]} \Rightarrow \overline{E}[f]$	$\underline{E}[f] \Rightarrow \underline{E[f]}$	$\overline{E \text{ rs } a} \Rightarrow \overline{E} \text{ rs } a$
$\underline{E} \text{ rs } a \Rightarrow \underline{E \text{ rs } a}$	$\overline{E \text{ sy } a} \Rightarrow \overline{E} \text{ sy } a$	$\underline{E} \text{ sy } a \Rightarrow \underline{E \text{ sy } a}$
$\overline{[E*F*K]} \Rightarrow [\overline{E}*F*K]$	$[\underline{E}*F*K] \Rightarrow [E*\overline{F}*K]$	$[E*\underline{F}*K] \Rightarrow [E*\overline{F}*K]$
$[E*\underline{F}*K] \Rightarrow [E*F*\overline{K}]$	$[E*F*\underline{K}] \Rightarrow \underline{[E*F*K]}$	

Let $E, F \in RegStatExpr,\ G, H, \widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$.

Inaction rules for arbitrary regular dynamic expressions

$$\frac{G \Rightarrow \widetilde{G},\ \circ \in \{;, []\}}{G \circ E \Rightarrow \widetilde{G} \circ E} \quad \frac{G \Rightarrow \widetilde{G},\ \circ \in \{;, []\}}{E \circ G \Rightarrow E \circ \widetilde{G}} \quad \frac{G \Rightarrow \widetilde{G}}{G \| H \Rightarrow \widetilde{G} \| H} \quad \frac{H \Rightarrow \widetilde{H}}{G \| H \Rightarrow G \| \widetilde{H}} \quad \frac{G \Rightarrow \widetilde{G}}{G[f] \Rightarrow \widetilde{G}[f]}$$

$$\frac{G \Rightarrow \widetilde{G},\ \circ \in \{\mathrm{rs}, \mathrm{sy}\}}{G \circ a \Rightarrow \widetilde{G} \circ a} \quad \frac{G \Rightarrow \widetilde{G}}{[G * E * F] \Rightarrow [\widetilde{G} * E * F]} \quad \frac{G \Rightarrow \widetilde{G}}{[E * G * F] \Rightarrow [E * \widetilde{G} * F]} \quad \frac{G \Rightarrow \widetilde{G}}{[E * F * G] \Rightarrow [E * F * \widetilde{G}]}$$

Definition 5 *A regular dynamic expression is operative if no inaction rule can be applied to it.*

$OpRegDynExpr$ is the set of *all operative regular dynamic expressions* of $dtsPBC$.

We shall consider regular expressions only and omit the word "regular".

Definition 6 $\approx\ = (\Rightarrow \cup \Leftarrow)^*$ *is the structural equivalence of dynamic expressions in* $dtsPBC$. G *and* G' *are structurally equivalent,* $G \approx G'$*, if they can be reached each from other by applying inaction rules in a forward or backward direction.*

Action and empty loop rules

Action rules: execution of non-empty multisets of activities at a time step.

Empty loop rule: execution of the empty multiset of activities at a time step.

Let $(\alpha,\rho),(\beta,\chi)\in\mathcal{SL},\ E,F\in RegStatExpr,\ G,H\in OpRegDynExpr,$ $\widetilde{G},\widetilde{H}\in RegDynExpr,\ a\in Act$ and $\Gamma,\Delta\in I\!N_{fin}^{\mathcal{SL}}\setminus\{\emptyset\},\ \Gamma'\in I\!N_{fin}^{\mathcal{SL}}$.

Action and empty loop rules

El $G\stackrel{\emptyset}{\rightarrow}G$

B $\overline{(\alpha,\rho)}\stackrel{\{(\alpha,\rho)\}}{\longrightarrow}\underline{(\alpha,\rho)}$

SC1 $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ \circ\in\{;,[]\}}{G\circ E\stackrel{\Gamma}{\rightarrow}\widetilde{G}\circ E}$

SC2 $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ \circ\in\{;,[]\}}{E\circ G\stackrel{\Gamma}{\rightarrow}E\circ\widetilde{G}}$

P1 $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{G\|H\stackrel{\Gamma}{\rightarrow}\widetilde{G}\|H}$

P2 $\dfrac{H\stackrel{\Gamma}{\rightarrow}\widetilde{H}}{G\|H\stackrel{\Gamma}{\rightarrow}G\|\widetilde{H}}$

P3 $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ H\stackrel{\Delta}{\rightarrow}\widetilde{H}}{G\|H\stackrel{\Gamma+\Delta}{\longrightarrow}\widetilde{G}\|\widetilde{H}}$

L $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{G[f]\stackrel{f(\Gamma)}{\longrightarrow}\widetilde{G}[f]}$

Rs $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ a,\hat{a}\not\in\mathcal{A}(\Gamma)}{G\ \text{rs}\ a\stackrel{\Gamma}{\rightarrow}\widetilde{G}\ \text{rs}\ a}$

I1 $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{[G*E*F]\stackrel{\Gamma}{\rightarrow}[\widetilde{G}*E*F]}$

I2 $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{[E*G*F]\stackrel{\Gamma}{\rightarrow}[E*\widetilde{G}*F]}$

I3 $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{[E*F*G]\stackrel{\Gamma}{\rightarrow}[E*F*\widetilde{G}]}$

Sy1 $\dfrac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{G\ \text{sy}\ a\stackrel{\Gamma}{\rightarrow}\widetilde{G}\ \text{sy}\ a}$

Sy2 $\dfrac{G\ \text{sy}\ a\xrightarrow{\Gamma'+\{(\alpha,\rho)\}+\{(\beta,\chi)\}}\widetilde{G}\ \text{sy}\ a,\ a\in\alpha,\ \hat{a}\in\beta}{G\ \text{sy}\ a\xrightarrow{\Gamma'+\{(\alpha\oplus_a\beta,\rho\cdot\chi)\}}\widetilde{G}\ \text{sy}\ a}$

Comparison of inaction, action and empty loop rules

Rules	State change	Time progress	Activities execution
Inaction rules	−	−	−
Action rules	±	+	+
Empty loop rule	−	+	−

Transition systems

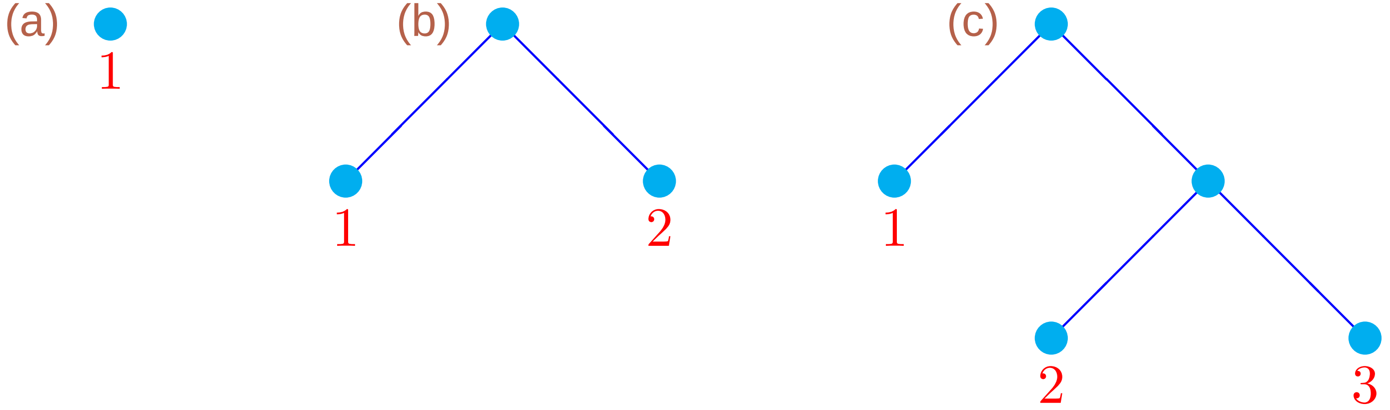
Definition 7 *Let $n \in \mathbb{N}$. The numbering of expressions is*

$$\iota ::= \ n \mid (\iota)(\iota).$$

Num is the set of *all numberings* of expressions.

The *content* of a numbering $\iota \in Num$ is

$$Cont(\iota) = \begin{cases} \{\iota\}, & \iota \in \mathbb{N}; \\ Cont(\iota_1) \cup Cont(\iota_2), & \iota = (\iota_1)(\iota_2). \end{cases}$$



BTRNUM:The binary trees encoded with the numberings 1, $(1)(2)$ and $(1)((2)(3))$

$[G]_\approx = \{H \mid G \approx H\}$ is the equivalence class of $G \in RegDynExpr$ w.r.t. structural equivalence.

Definition 8 *The derivation set $DR(G)$ of a dynamic expression G is the minimal set:*

- $[G]_\approx \in DR(G)$*;*
- *if* $[H]_\approx \in DR(G)$ *and* $\exists \Gamma\ H \stackrel{\Gamma}{\rightarrow} \widetilde{H}$ *then* $[\widetilde{H}]_\approx \in DR(G)$*.*

Let G be a dynamic expression and $s, \tilde{s} \in DR(G)$.

The set of *all multisets of activities executable from s* is $Exec(s) = \{\Gamma \mid \exists H \in s\ \exists \widetilde{H}\ H \stackrel{\Gamma}{\rightarrow} \widetilde{H}\}$.

Let $\Gamma \in Exec(s) \setminus \{\emptyset\}$. The *probability that the multiset of activities Γ is ready for execution in s*:

$$PF(\Gamma, s) = \prod_{(\alpha,\rho)\in\Gamma} \rho \cdot \prod_{\{\{(\beta,\chi)\}\in Exec(s) \mid (\beta,\chi)\notin\Gamma\}} (1-\chi).$$

In the case $\Gamma = \emptyset$ we define

$$PF(\emptyset, s) = \begin{cases} \prod_{\{(\beta,\chi)\}\in Exec(s)} (1-\chi), & Exec(s) \neq \{\emptyset\}; \\ 1, & \text{otherwise}. \end{cases}$$

Let $\Gamma \in Exec(s)$. The *probability to execute the multiset of activities* Γ *in* s:

$$PT(\Gamma, s) = \frac{PF(\Gamma, s)}{\sum_{\Delta \in Exec(s)} PF(\Delta, s)}.$$

The *probability to move from* s *to* $\tilde{s}$ *by executing any multiset of activities*:

$$PM(s, \tilde{s}) = \sum_{\{\Gamma \mid \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \xrightarrow{\Gamma} \widetilde{H}\}} PT(\Gamma, s).$$

Calculation of the probability functions PF, PT, PM for $s_1 \in DR(\overline{E})$ and $E = (\{a\}, \rho)[](\{a\}, \chi)$

$s_1 \backslash \Gamma$	$\emptyset$	$\{(\{a\}, \rho)\}$	$\{(\{a\}, \chi)\}$	Σ
PF	$(1-\rho)(1-\chi)$	$\rho(1-\chi)$	$\chi(1-\rho)$	$1-\rho\chi$
PT	$\frac{(1-\rho)(1-\chi)}{1-\rho\chi}$	$\frac{\rho(1-\chi)}{1-\rho\chi}$	$\frac{\chi(1-\rho)}{1-\rho\chi}$	1
PM	$\frac{(1-\rho)(1-\chi)}{1-\rho\chi}\ (s_1)$	$\frac{\rho+\chi-2\rho\chi}{1-\rho\chi}\ (s_2)$		1

Definition 9 *The* (labeled probabilistic) transition system *of a dynamic expression* G *is* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, *where*

- *the set of* states *is* $S_G = DR(G)$;
- *the set of* labels *is* $L_G = I\!N_{fin}^{\mathcal{SL}} \times (0; 1]$;
- *the set of* transitions *is*
  $\mathcal{T}_G = \{(s, (\Gamma, PT(\Gamma, s)), \tilde{s}) \mid s, \tilde{s} \in DR(G),\ \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \stackrel{\Gamma}{\to} \widetilde{H}\}$;
- *the* initial state *is* $s_G = [G]_\approx$.

A transition $(s, (\Gamma, \mathcal{P}), \tilde{s}) \in \mathcal{T}_G$ is written as $s \stackrel{\Gamma}{\to}_{\mathcal{P}} \tilde{s}$.

We write $s \stackrel{\Gamma}{\to} \tilde{s}$ if $\exists \mathcal{P}\ s \stackrel{\Gamma}{\to}_{\mathcal{P}} \tilde{s}$ and $s \to \tilde{s}$ if $\exists \Gamma\ s \stackrel{\Gamma}{\to} \tilde{s}$.

Definition 10 *Let* G, G' *be dynamic expressions and* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, $TS(G') = (S_{G'}, L_{G'}, \mathcal{T}_{G'}, s_{G'})$ *be their transition systems. A mapping* $\beta : S_G \rightarrow S_{G'}$ *is an* isomorphism *between* $TS(G)$ *and* $TS(G')$, $\beta : TS(G) \simeq TS(G')$, *if*

1. β *is a bijection s.t.* $\beta(s_G) = s_{G'}$;
2. $\forall s, \tilde{s} \in S_G \ \forall \Gamma \ s \stackrel{\Gamma}{\rightarrow}_{\mathcal{P}} \tilde{s} \ \Leftrightarrow \ \beta(s) \stackrel{\Gamma}{\rightarrow}_{\mathcal{P}} \beta(\tilde{s})$.

$TS(G)$ *and* $TS(G')$ *are* isomorphic, $TS(G) \simeq TS(G')$, *if* $\exists \beta : TS(G) \simeq TS(G')$.

For $E \in RegStatExpr$, let $TS(E) = TS(\overline{E})$.

Definition 11 G *and* G' *are* equivalent w.r.t. transition systems, $G =_{ts} G'$, *if* $TS(G) \simeq TS(G')$.

For a dynamic expression G, a discrete random variable is associated with every state $s \in DR(G)$.

The random variables (residence time in the states) are geometrically distributed:
the probability to stay in the state $s \in DR(G)$ for $k-1$ moments and leave it at the moment $k \geq 1$ is $PM(s,s)^{k-1}(1-PM(s,s))$.

The mean value formula: the *average sojourn time in the state* s is

$$SJ(s) = \frac{1}{1-PM(s,s)}.$$

The *average sojourn time vector* SJ of G has the elements $SJ(s),\ s \in DR(G)$.

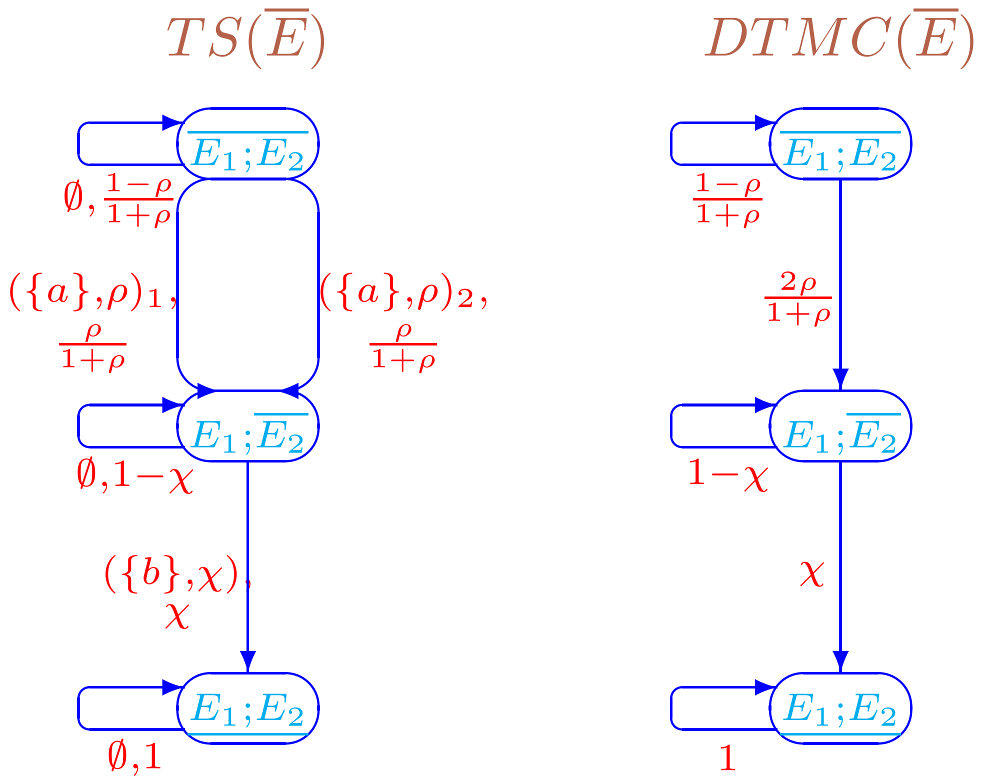
Analogously: the *sojourn time variance in the state* s is

$$VAR(s) = \frac{PM(s,s)}{(1-PM(s,s))^2}.$$

The *sojourn time variance vector* VAR of G has the elements $VAR(s),\ s \in DR(G)$.

Definition 12 *The* underlying discrete time Markov chain (DTMC) *of a dynamic expression* G*,* $DTMC(G)$*, has the state space* $DR(G)$*, the initial state* $[G]_{\approx}$ *and transitions* $s \rightarrow_{\mathcal{P}} \tilde{s}$*, if* $s \rightarrow \tilde{s}$ *and* $\mathcal{P} = PM(s, \tilde{s})$*.*

For $E \in RegStatExpr$, let $DTMC(E) = DTMC(\overline{E})$.

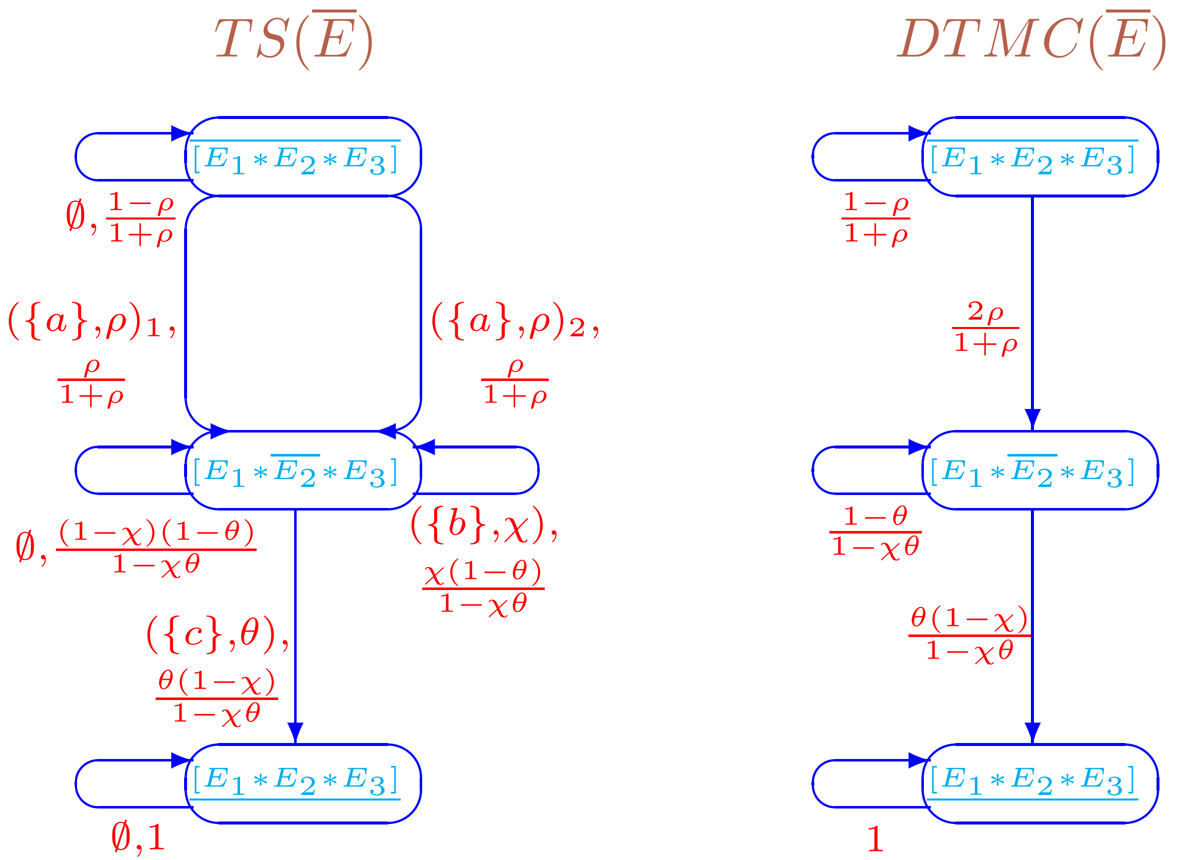


The transition system and the underlying DTMC of $\overline{E}$ for $E = ((\{a\},\rho)_1[](\{a\},\rho)_2);(\{b\},\chi)$

Let $E_1 = (\{a\},\rho)[](\{a\},\rho),\ E_2 = (\{b\},\chi)$ and $E = E_1;E_2$.

The identical activities of the composite static expression are enumerated as:

$E = ((\{a\},\rho)_1[](\{a\},\rho)_2);(\{b\},\chi)$.



EXPRIT:The transition system and the underlying DTMC of $\overline{E}$ for $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$

Let $E_1 = (\{a\},\rho)[](\{a\},\rho),\ E_2 = (\{b\},\chi),\ E_3 = (\{c\},\theta)$ and $E = [E_1 * E_2 * E_3]$.

The identical activities of the composite static expression are enumerated as:

$E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$.

$DR(\overline{E})$ consists of $s_1 = [\overline{[E_1 * E_2 * E_3]}]_\approx,\ s_2 = [[E_1 * \overline{E_2} * E_3]]_\approx,\ s_3 = [\underline{[E_1 * E_2 * E_3]}]_\approx$.

The average sojourn time vector of $\overline{E}$ is

$$SJ = \left(\frac{1+\rho}{2\rho}, \frac{1-\chi\theta}{\theta(1-\chi)}, \infty\right).$$

The sojourn time variance vector of $\overline{E}$ is

$$VAR = \left(\frac{1-\rho^2}{4\rho^2}, \frac{(1-\theta)(1-\chi\theta)}{\theta^2(1-\chi)^2}, \infty\right).$$

Denotational semantics

Labeled DTSPNs

Definition 13 *A* labeled discrete time stochastic Petri net (LDTSPN) *is* $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$*, where*

- P_N *and* T_N *are finite sets of* places *and* transitions *(*$P_N \cup T_N \neq \emptyset,\ P_N \cap T_N = \emptyset$*);*
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbb{N}$ *is the* arc weight *function;*
- $\Omega_N : T_N \to (0; 1)$ *is the* transition probability *function;*
- $L_N : T_N \to \mathcal{L}$ *is the* transition labeling *function;*
- $M_N \in \mathbb{N}_{fin}^{P_N}$ *is the* initial marking*.*

Concurrent transition firings at discrete time moments.

LDTSPNs have *step* semantics.

A transition $t \in T_N$ is *enabled* in a marking $M \in I\!N_{fin}^{P_N}$ of LDTSPN N if ${}^\bullet t \subseteq M$.

$Ena(M)$ is the set of *all transitions enabled in* M.

A set of transitions $U \subseteq Ena(M)$ is *enabled* in M if ${}^\bullet U \subseteq M$.

Then $t \in Ena(M)$ fires in the next time moment with probability $\Omega_N(t)$, if no different transition is enabled in M, i.e. $Ena(M) = \{t\}$.

Let $U \subseteq Ena(M),\ U \neq \emptyset$ and ${}^\bullet U \subseteq M$. The *probability that the set of transitions* U *is ready for firing in* M:

$$PF(U, M) = \prod_{t \in U} \Omega_N(t) \cdot \prod_{u \in Ena(M) \setminus U} (1 - \Omega_N(u)).$$

In the case $U = \emptyset$ we define

$$PF(\emptyset, M) = \begin{cases} \prod_{u \in Ena(M)} (1 - \Omega_N(u)) & Ena(M) \neq \emptyset; \\ 1 & \text{otherwise}. \end{cases}$$

Let $U \subseteq Ena(M)$ and ${}^\bullet U \subseteq M$. The *probability that the set of transitions U fires in M*:

$$PT(U, M) = \frac{PF(U, M)}{\sum_{\{V \subseteq Ena(M) \mid {}^\bullet V \subseteq M\}} PF(V, M)}.$$

If $U = \emptyset$ then $M = \widetilde{M}$.

Firing of U changes marking M to $\widetilde{M} = M - {}^\bullet U + U^\bullet$, $M \stackrel{U}{\rightarrow}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PT(U, M)$.

We write $M \stackrel{U}{\rightarrow} \widetilde{M}$ if $\exists \mathcal{P}\ M \stackrel{U}{\rightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \rightarrow \widetilde{M}$ if $\exists U\ M \stackrel{U}{\rightarrow} \widetilde{M}$.

For $U = \{t\}$ we write $M \stackrel{t}{\rightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \stackrel{t}{\rightarrow} \widetilde{M}$.

Definition 14 *Let N be an LDTSPN.*

- *The* reachability set $RS(N)$ *is the minimal set of markings s.t.*
  - $M_N \in RS(N)$;
  - *if* $M \in RS(N)$ *and* $M \rightarrow \widetilde{M}$ *then* $\widetilde{M} \in RS(N)$.
- *The* reachability graph $RG(N)$ *is a directed labeled graph with*
  - *the set of nodes* $RS(N)$;
  - *an arc labeled by* $(U, \mathcal{P})$ *from node* M *to* $\widetilde{M}$ *if* $M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$.
- *The* underlying Discrete Time Markov Chain (DTMC) $DTMC(N)$ *is a DTMC with*
  - *the state space* $RS(N)$;
  - *a transition* $M \rightarrow_{\mathcal{P}} \widetilde{M}$, *where* $\mathcal{P} = PM(M, \widetilde{M})$ *is the* probability to move from M to $\widetilde{M}$ by firing any set of transitions*:*

$$PM(M, \widetilde{M}) = \sum_{\{U \mid M \xrightarrow{U} \widetilde{M}\}} PT(U, M);$$

  - *the initial state* M_N.

Let N be an LDTSPN and $M \in RS(N)$. The *average sojourn time in the marking* M is

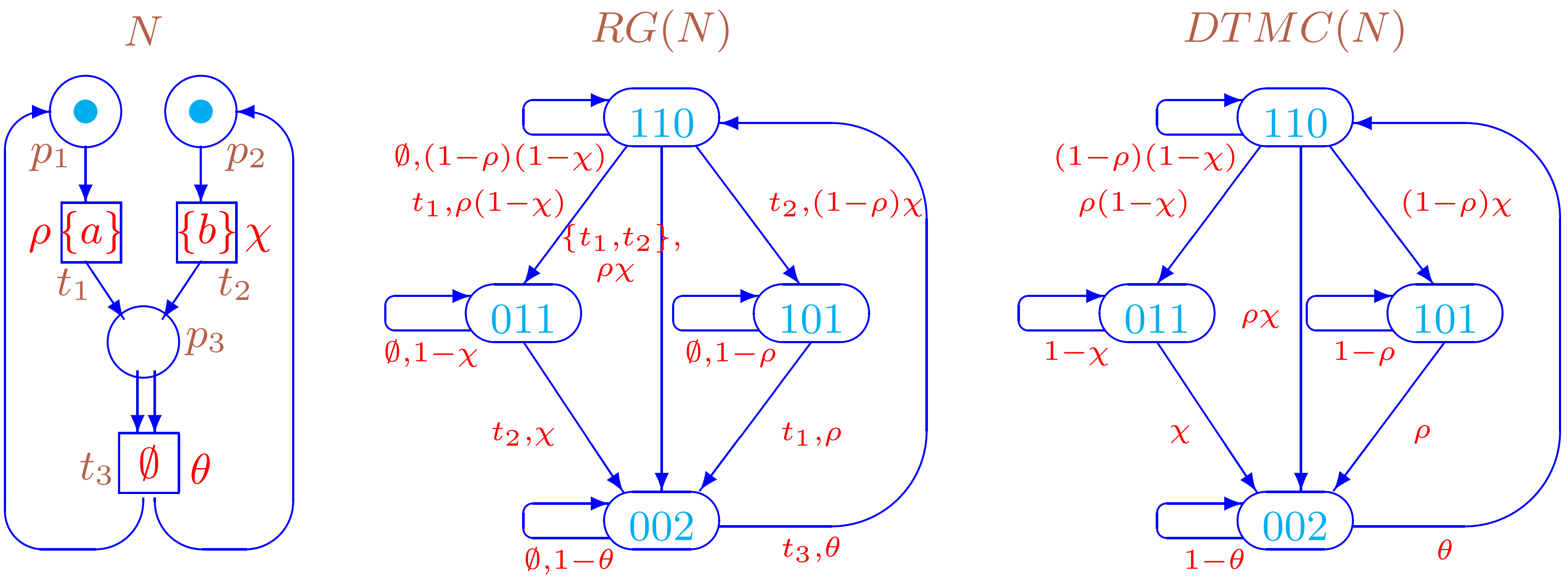
$$SJ(M) = \frac{1}{1 - PM(M, M)}.$$

The *average sojourn time vector* SJ of N has the elements $SJ(M),\ M \in RS(N)$.

The *sojourn time variance in the marking* M is

$$VAR(M) = \frac{PM(M, M)}{(1 - PM(M, M))^2}.$$

The *sojourn time variance vector* VAR of N has the elements $VAR(M),\ M \in RS(N)$.



LDTSPN, its reachability graph and the underlying DTMC

The transitions: t_1 (labeled by $\{a\}$), t_2 (labeled by $\{b\}$) and t_3 (labeled by $\emptyset$).

The transition probabilities: $\rho = \Omega_N(t_1),\ \chi = \Omega_N(t_2),\ \theta = \Omega_N(t_3)$.

$RS(N)$ consists of $M_1 = (1,1,0),\ M_2 = (0,1,1),\ M_3 = (1,0,1),\ M_4 = (0,0,2)$.

The average sojourn time vector of N:

$$SJ = \left(\frac{1}{\rho+\chi-\rho\chi}, \frac{1}{\chi}, \frac{1}{\rho}, \frac{1}{\theta}\right).$$

The sojourn time variance vector of N:

$$VAR = \left(\frac{1-\rho-\chi+\rho\chi}{(\rho+\chi-\rho\chi)^2}, \frac{1-\chi}{\chi^2}, \frac{1-\rho}{\rho^2}, \frac{1-\theta}{\theta^2}\right).$$

The elements $\mathcal{P}_{ij}(1 \leq i, j \leq 4)$ of (one-step) transition probability matrix (TPM) of $DTMC(N)$:

$$\mathcal{P}_{ij} = \begin{cases} PM(s_i, s_j) & s_i \to s_j; \\ 0 & \text{otherwise}. \end{cases}$$

The (one-step) TPM:

$$\mathbf{P} = \begin{pmatrix} (1-\rho)(1-\chi) & \rho(1-\chi) & \chi(1-\rho) & \rho\chi \\ 0 & 1-\chi & 0 & \chi \\ 0 & 0 & 1-\rho & \rho \\ \theta & 0 & 0 & 1-\theta \end{pmatrix}$$

The steady-state PMF ψ is a solution of

$$\begin{cases} \psi(\mathbf{P}-\mathbf{I}) = \mathbf{0} \\ \psi\mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{I}$ is the identity matrix of size four and $\mathbf{0} = (0, 0, 0, 0)$, $\mathbf{1} = (1, 1, 1, 1)$.

For $\rho = \chi = \theta$

$$\psi = \left(\frac{1}{5-3\rho}, \frac{1-\rho}{5-3\rho}, \frac{1-\rho}{5-3\rho}, \frac{2-\rho}{5-3\rho}\right).$$

The inverse of the steady-state PMF is the mean recurrence time vector

$$RC = \left(5-3\rho, \frac{5-3\rho}{1-\rho}, \frac{5-3\rho}{1-\rho}, \frac{5-3\rho}{2-\rho}\right).$$

The average time to come back to the initial marking $M_N = M_1$ in the long-term behaviour is in $(2;5)$.

Algebra of dts-boxes

Definition 15 *A* discrete time stochastic Petri box (dts-box) *is* $N = (P_N, T_N, W_N, \Lambda_N)$, *where*

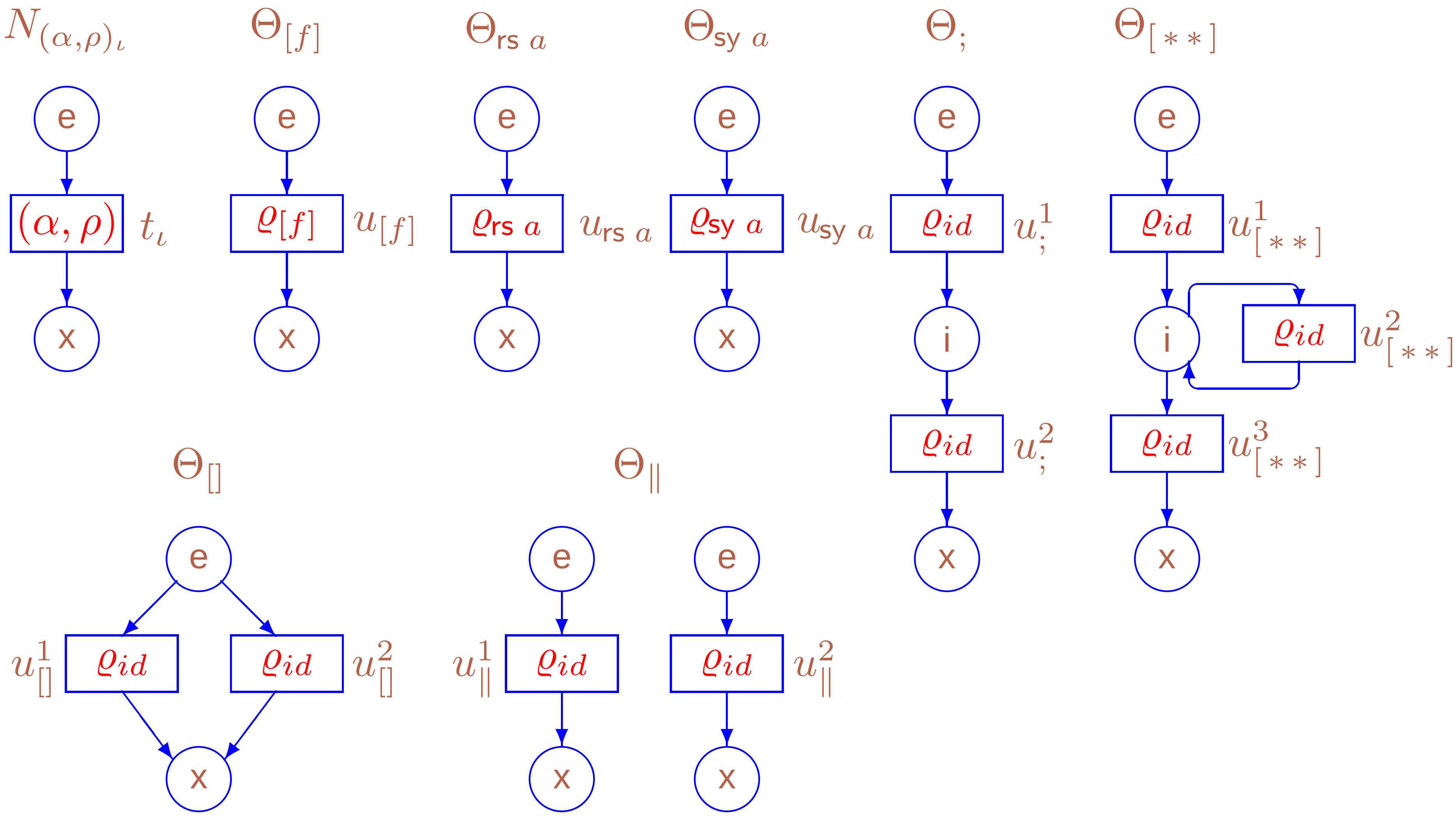
- P_N *and* T_N *are finite sets of* places *and* transitions, *respectively, s.t.* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$;
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to I\!N$ *is a function of the* weights of arcs *between places and transitions and vice versa;*
- Λ_N *is the* place and transition labeling *function s.t.*
  - $\Lambda_N|_{P_N} : P_N \to \{\mathsf{e}, \mathsf{i}, \mathsf{x}\}$ *(it specifies* entry, internal *and* exit *places);*
  - $\Lambda_N|_{T_N} : T_N \to \{\varrho \mid \varrho \subseteq I\!N_{fin}^{\mathcal{SL}} \times \mathcal{SL}\}$ *(it associates transitions with the* relabeling relations*).*

Moreover, $\forall t \in T_N\ {}^\bullet t \neq \emptyset \neq t^\bullet$.

For the set of entry *places of* N, ${}^\circ N = \{p \in P_N \mid \Lambda_N(p) = \mathsf{e}\}$, *and the set of* exit *places of* N, $N^\circ = \{p \in P_N \mid \Lambda_N(p) = \mathsf{x}\}$, *it holds:* ${}^\circ N \neq \emptyset \neq N^\circ$ *and* ${}^\bullet({}^\circ N) = \emptyset = (N^\circ)^\bullet$.

A dts-box is *plain* if $\forall t \in T_N\ \Lambda_N(t) = \varrho_{(\alpha,\rho)}$, where $\varrho_{(\alpha,\rho)} = \{(\emptyset, (\alpha, \rho))\}$ is the constant relabeling, identified with (α, ρ).

A *marked plain dts-box* is a pair (N, M_N), where N is a plain dts-box and $M_N \in I\!N_{fin}^{P_N}$ is its marking. Let $\overline{N} = (N, {}^\circ N)$ and $\underline{N} = (N, N^\circ)$.



The plain and operator dts-boxes

Definition 16 *Let* $(\alpha, \rho) \in \mathcal{SL},\ a \in Act$ *and* $E, F, K \in RegStatExpr$. *The* denotational semantics *of* $dtsPBC$ *is a mapping* Box_{dts} *from* $RegStatExpr$ *into plain dts-boxes:*

1. $Box_{dts}((\alpha, \rho)_\iota) = N_{(\alpha,\rho)_\iota}$;
2. $Box_{dts}(E \circ F) = \Theta_\circ(Box_{dts}(E), Box_{dts}(F)),\ \circ \in \{;, [], \|\}$;
3. $Box_{dts}(E[f]) = \Theta_{[f]}(Box_{dts}(E))$;
4. $Box_{dts}(E \circ a) = \Theta_{\circ a}(Box_{dts}(E)),\ \circ \in \{\mathsf{rs}, \mathsf{sy}\}$;
5. $Box_{dts}([E * F * K]) = \Theta_{[\,*\,*\,]}(Box_{dts}(E), Box_{dts}(F), Box_{dts}(K))$.

For $E \in RegStatExpr$, let $Box_{dts}(\overline{E}) = \overline{Box_{dts}(E)}$ and $Box_{dts}(\underline{E}) = \underline{Box_{dts}(E)}$.

We denote isomorphism of transition systems by $\simeq$,

and the same symbol denotes isomorphism of reachability graphs and DTMCs

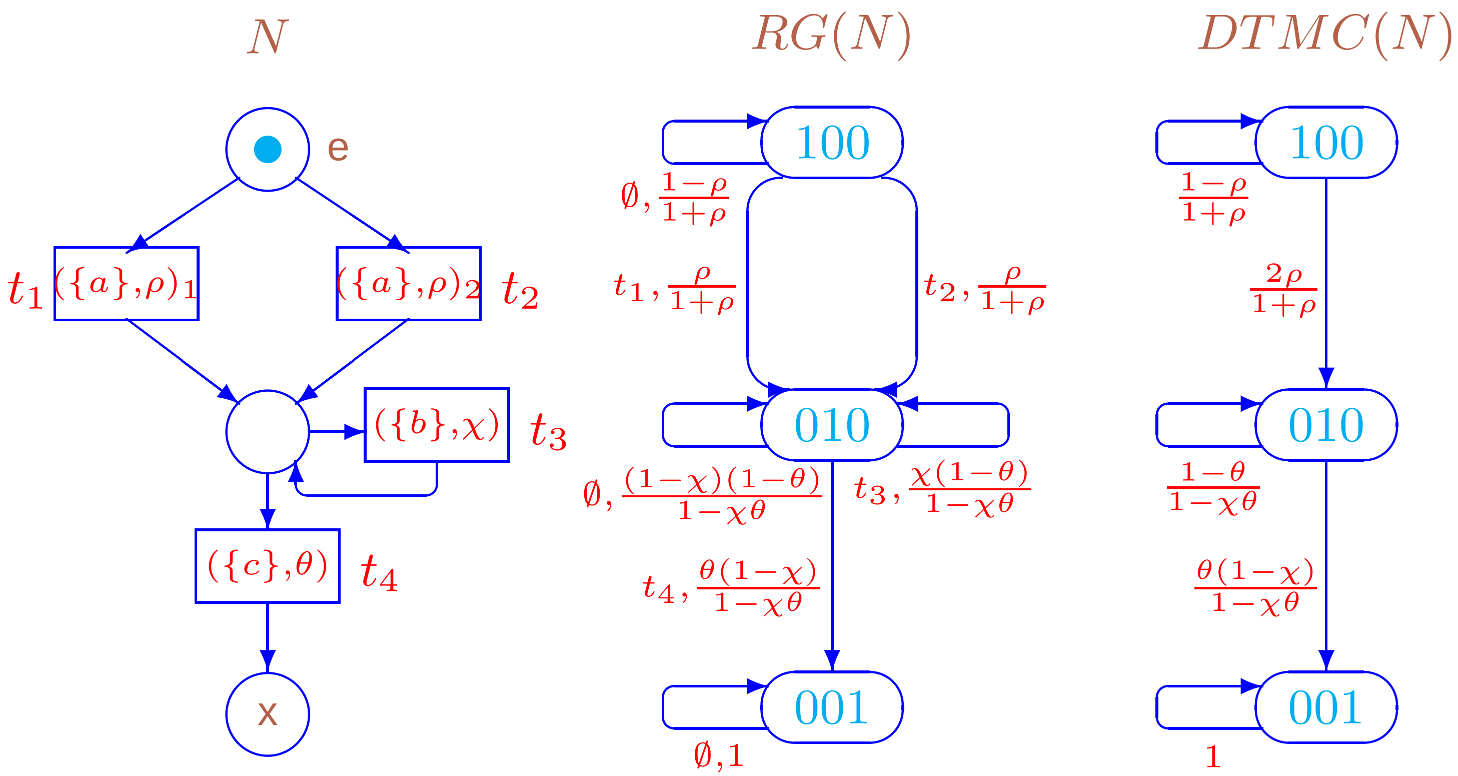
as well as isomorphism between transition systems and reachability graphs.

Theorem 1 *For any static expression* E

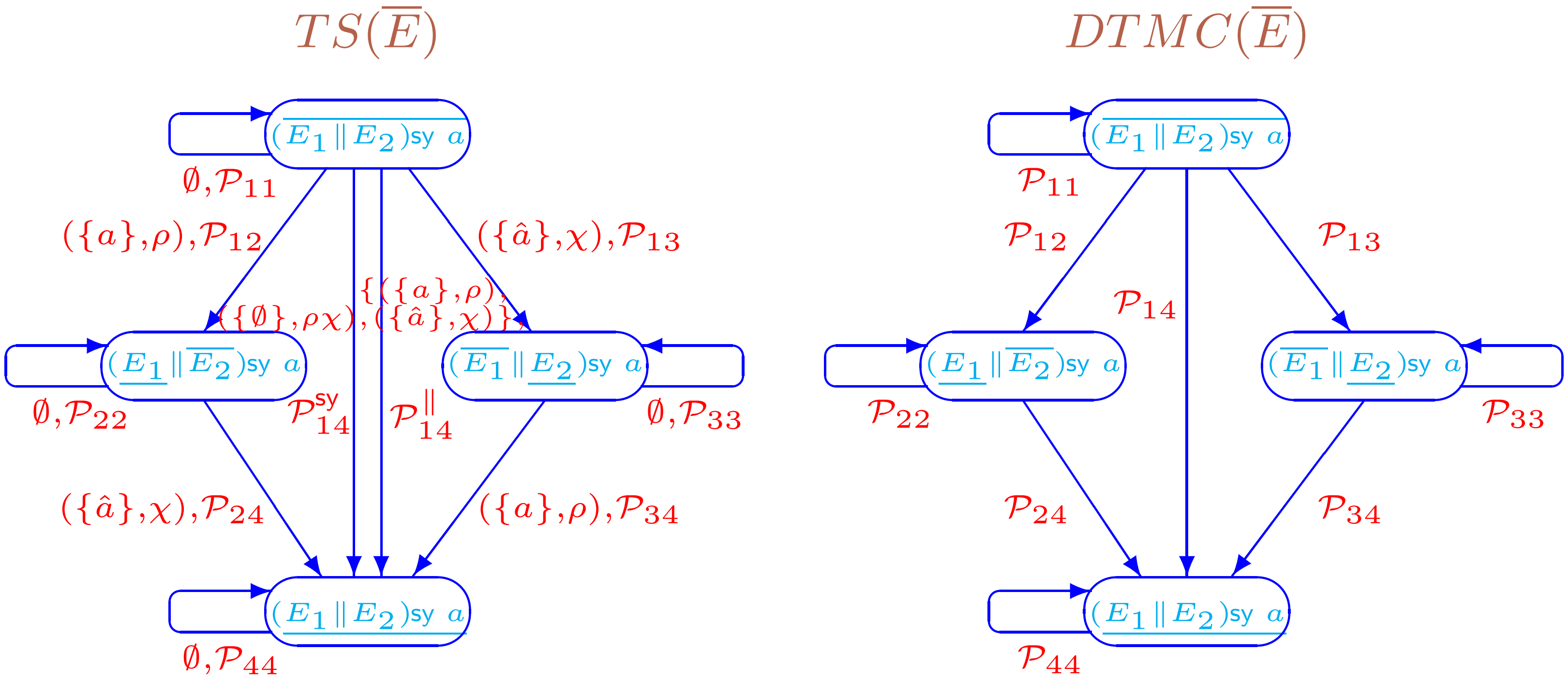
$$TS(\overline{E}) \simeq RG(Box_{dts}(\overline{E})).$$

Proposition 1 *For any static expression* E

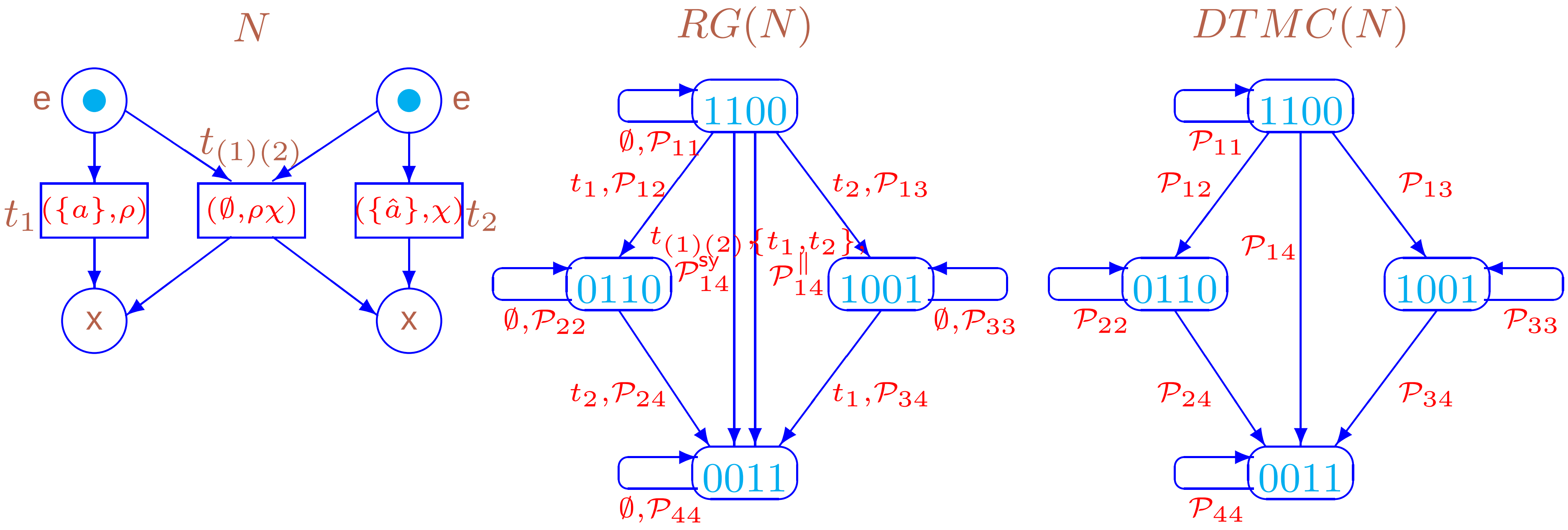
$$DTMC(\overline{E}) \simeq DTMC(Box_{dts}(\overline{E})).$$



BOXIT:The marked dts-box $N = Box_{dts}(\overline{E})$ for $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$, its reachability graph and the underlying DTMC



EXPR:The transition system and the underlying DTMC of $\overline{E}$ for $E = ((\{a\},\rho)\|(\{\hat{a}\},\chi))$ sy a



BOX:The marked dts-box $N = Box_{dts}(\overline{E})$ for $E = ((\{a\},\rho)\|(\{\hat{a}\},\chi))$ sy a, its reachability graph and the underlying DTMC

The normalization factor $\mathcal{N} = \frac{1}{1-\rho^2\chi-\rho\chi^2+\rho^2\chi^2}$.

$$\mathcal{P}_{11} = \mathcal{N}(1-\rho)(1-\chi)(1-\rho\chi) \qquad \mathcal{P}_{12} = \mathcal{N}\rho(1-\chi)(1-\rho\chi)$$

$$\mathcal{P}_{13} = \mathcal{N}\chi(1-\rho)(1-\rho\chi) \qquad \mathcal{P}_{14}^{\mathsf{sy}} = \mathcal{N}\rho\chi(1-\rho)(1-\chi)$$

$$\mathcal{P}_{14}^{\|} = \mathcal{N}\rho\chi(1-\rho\chi) \qquad \mathcal{P}_{22} = 1-\chi$$

$$\mathcal{P}_{24} = \chi \qquad \mathcal{P}_{33} = 1-\rho$$

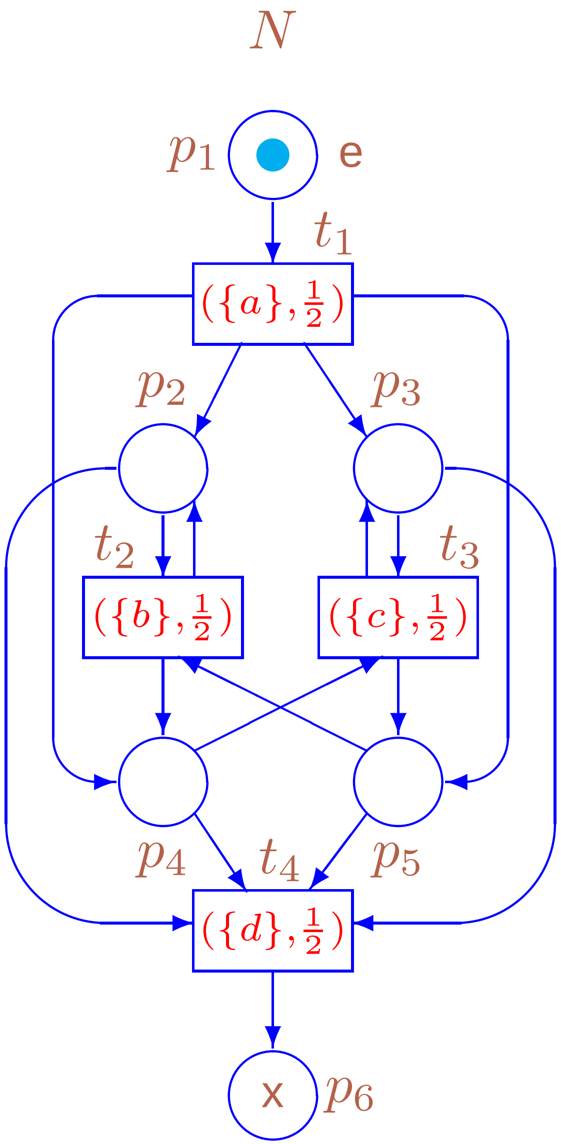
$$\mathcal{P}_{34} = \rho \qquad \mathcal{P}_{44} = 1$$

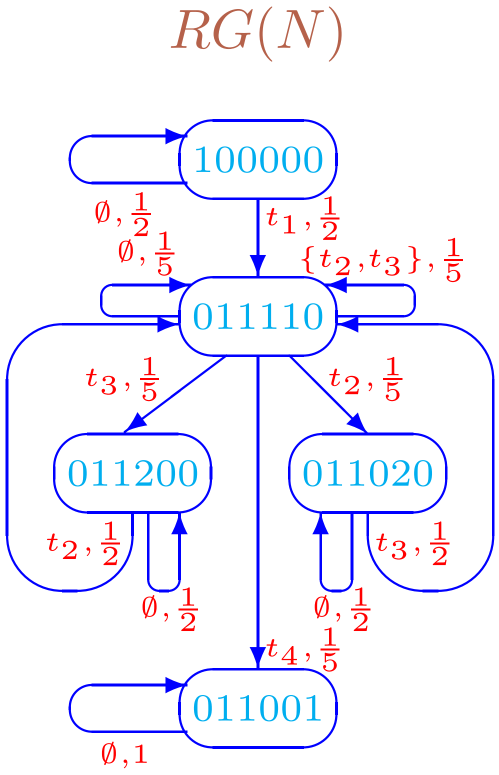
$$\mathcal{P}_{14} = \mathcal{P}_{14}^{\mathsf{sy}} + \mathcal{P}_{14}^{\|} = \mathcal{N}\rho\chi(2-\rho-\chi)$$

The case $\rho = \chi = \frac{1}{2}$:

$$\mathcal{P}_{11} = \mathcal{P}_{12} = \mathcal{P}_{13} = \mathcal{P}_{14}^{\|} = \frac{3}{13},\ \mathcal{P}_{14}^{\mathsf{sy}} = \frac{1}{13},$$

$$\mathcal{P}_{22} = \mathcal{P}_{24} = \mathcal{P}_{33} = \mathcal{P}_{34} = \frac{1}{2},\ \mathcal{P}_{44} = 1,\ \mathcal{P}_{14} = \frac{4}{13}.$$





The marked dts-box $N = Box_{dts}(\overline{E})$ for $E = [((\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}) \| (\{c\}, \frac{1}{2})) * (\{d\}, \frac{1}{2})]$

and its reachability graph

$M_1 = (1, 0, 0, 0, 0, 0)$ is the initial marking.

$M_2 = (0, 1, 1, 1, 1, 0)$ is obtained from M_1 by firing t_1.

$M_3 = (0, 1, 1, 2, 0, 0)$ is obtained from M_2 by firing t_2 and has 2 tokens in the place p_4.

$M_4 = (0, 1, 1, 0, 2, 0)$ is obtained from M_2 by firing t_3 and has 2 tokens in the place p_5.

Concurrency in the second argument of iteration in $\overline{E}$ can lead to non-safeness of the corresponding marked dts-box N, but it is 2-bounded in the worst case.

The origin of the problem: N has as a self-loop with two subnets which can function independently.

Stochastic equivalences

Empty loops in transition systems

Let G be a dynamic expression and $s \in DR(G)$.

The *probability to stay in s due to k $(k \geq 1)$ empty loops* is $(PT(\emptyset, s))^k$.

Let $\Gamma \in Exec(s) \setminus \{\emptyset\}$, i.e. $PT(\emptyset, s) < 1$. The *probability to execute the non-empty multiset of activities Γ in s after possible empty loops*:

$$PT^*(\Gamma, s) = PT(\Gamma, s) \sum_{k=0}^{\infty} (PT(\emptyset, s))^k = \frac{PT(\Gamma, s)}{1 - PT(\emptyset, s)} = EL(s) PT(\Gamma, s),$$

where $EL(s) = \frac{1}{1 - PT(\emptyset, s)}$ is the *empty loops abstraction factor*.

The *empty loops abstraction vector* EL of G has the elements $EL(s),\ s \in DR(G)$.

Definition 17 *The* (labeled probabilistic) transition system without empty loops $TS^*(G)$ *has the state space* $DR(G)$ *and the transitions* $s \stackrel{\Gamma}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$, *if* $s \stackrel{\Gamma}{\rightarrow} \tilde{s},\ \Gamma \neq \emptyset$ *and* $\mathcal{P} = PT^*(\Gamma, s)$.

We write $s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}$ if $\exists \mathcal{P}\ s \stackrel{\Gamma}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$ and $s \twoheadrightarrow \tilde{s}$ if $\exists \Gamma\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}$.

For $\Gamma = \{(\alpha, \rho)\}$ we write $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$ and $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}$.

For $E \in RegStatExpr$, let $TS^*(E) = TS^*(\overline{E})$.

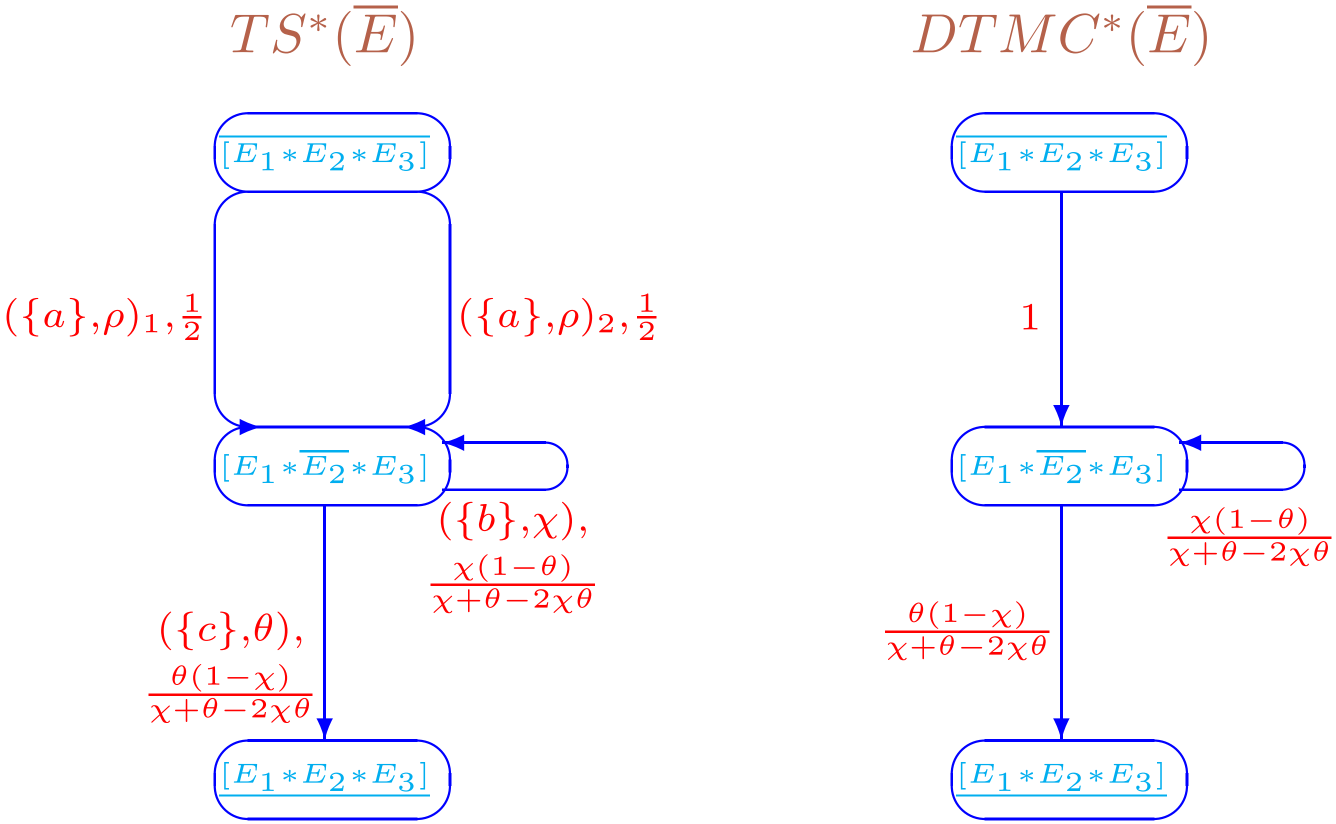
Definition 18 *G and G' are equivalent w.r.t. transition systems without empty loops, $G =_{ts*} G'$, if $TS^*(G) \simeq TS^*(G')$.*

Definition 19 *The underlying DTMC without empty loops $DTMC^*(G)$ has the state space $DR(G)$ and transitions $s \twoheadrightarrow_{\mathcal{P}} \tilde{s}$, if $s \twoheadrightarrow \tilde{s}$, where $\mathcal{P} = PM^*(s, \tilde{s})$ is the probability to move from s to $\tilde{s}$ by executing any non-empty multiset of activities after possible empty loops:*

$$PM^*(s,\tilde{s}) = \sum_{\{\Gamma \mid s \xrightarrow{\Gamma}\!\!\!\twoheadrightarrow \tilde{s}\}} PT^*(\Gamma, s) = \begin{cases} EL(s)(PM(s,s) - PT(\emptyset, s)), & s = \tilde{s}; \\ EL(s)PM(s,\tilde{s}), & \text{otherwise}, \end{cases}$$

where $PM(s,s) - PT(\emptyset, s)$ is the probability to stay in s due to any non-empty loop, i.e. by executing any non-empty multiset of activities.

For $E \in RegStatExpr$, let $DTMC^*(E) = DTMC^*(\overline{E})$.



The transition system and the underlying DTMC without empty loops of $\overline{E}$ in Figure EXPRIT

Empty loops in reachability graphs

Let N be an LDTSPN and $M \in RS(N)$.

The *probability to stay in M due to k $(k \geq 1)$ empty loops* is $(PT(\emptyset, M))^k$.

Let $U \subseteq Ena(M),\ U \neq \emptyset$ and ${}^\bullet U \subseteq M$, i.e. $PT(\emptyset, M) < 1$. The *probability that the non-empty set of transitions U fires in M after possible empty loops*:

$$PT^*(U, M) = PT(U, M) \sum_{k=0}^{\infty} (PT(\emptyset, M))^k = \frac{PT(U, M)}{1 - PT(\emptyset, M)} = EL(M) PT(U, M),$$

where $EL(M) = \frac{1}{1 - PT(\emptyset, M)}$ is the *empty loops abstraction factor*.

The *empty loops abstraction vector EL* of N has the elements $EL(M),\ M \in RS(N)$.

Definition 20 *The* reachability graph without empty loops $RG^*(N)$ *with the set of nodes* $RS(N)$ *and the set of arcs corresponding to the transitions* $M \stackrel{U}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$, *if* $M \stackrel{U}{\rightarrow} \widetilde{M},\ U \neq \emptyset$ *and* $\mathcal{P} = PT^*(U, M)$.

We write $M \stackrel{U}{\twoheadrightarrow} \widetilde{M}$ if $\exists \mathcal{P}\ M \stackrel{U}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \twoheadrightarrow \widetilde{M}$ if $\exists U\ M \stackrel{U}{\twoheadrightarrow} \widetilde{M}$.

For $U = \{t\}$ we write $M \stackrel{t}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \stackrel{t}{\twoheadrightarrow} \widetilde{M}$.

Definition 21 *The* underlying DTMC without empty loops $DTMC^*(N)$ *has the state space* $RS(N)$ *and transitions* $M \twoheadrightarrow_{\mathcal{P}} \widetilde{M}$, *if* $M \twoheadrightarrow \widetilde{M}$, *where* $\mathcal{P} = PM^*(M, \widetilde{M})$ *is the* probability to move from M to $\widetilde{M}$ by firing any non-empty set of transitions after possible empty loops*:*

$$PM^*(M, \widetilde{M}) = \sum_{\{U \in Ena(M) \mid M \stackrel{U}{\twoheadrightarrow} \widetilde{M}\}} PT^*(U, M) = \begin{cases} EL(M)(PM(M,M) - PT(\emptyset, M)), & M = \widetilde{M}; \\ EL(M)PM(M, \widetilde{M}), & \text{otherwise}, \end{cases}$$

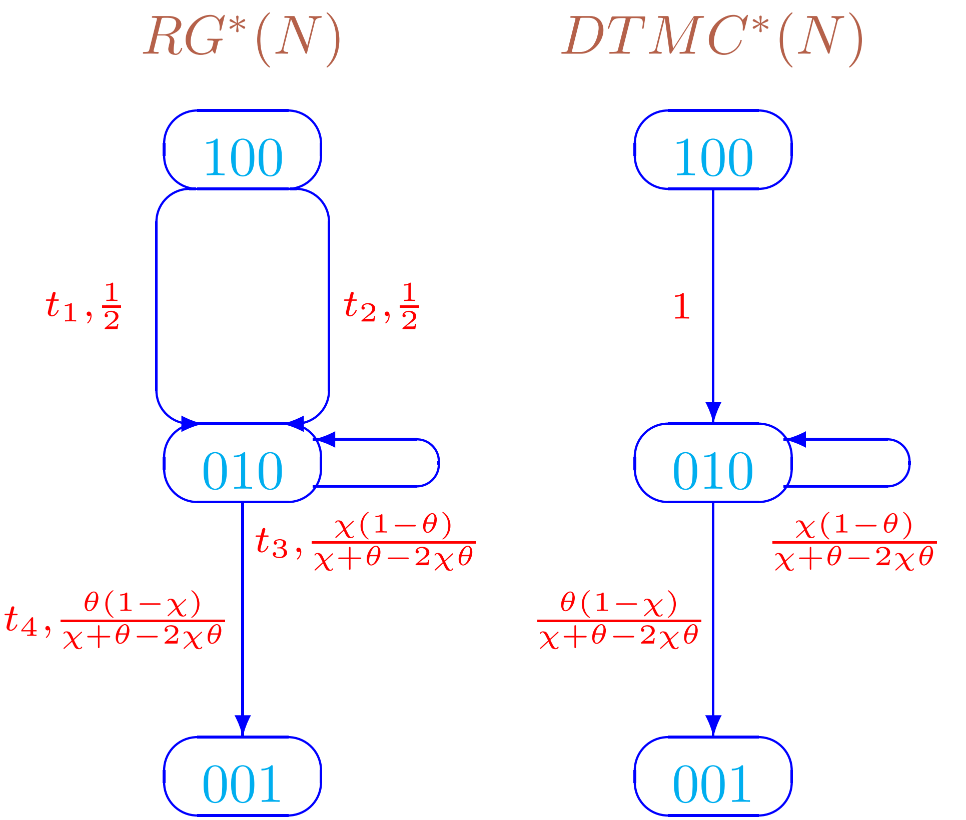
where $PM(M,M) - PT(\emptyset, M)$ *is the* probability to stay in M due to any non-empty loop, *i.e. by firing any non-empty multiset of transitions.*

Theorem 2 *For any static expression* E

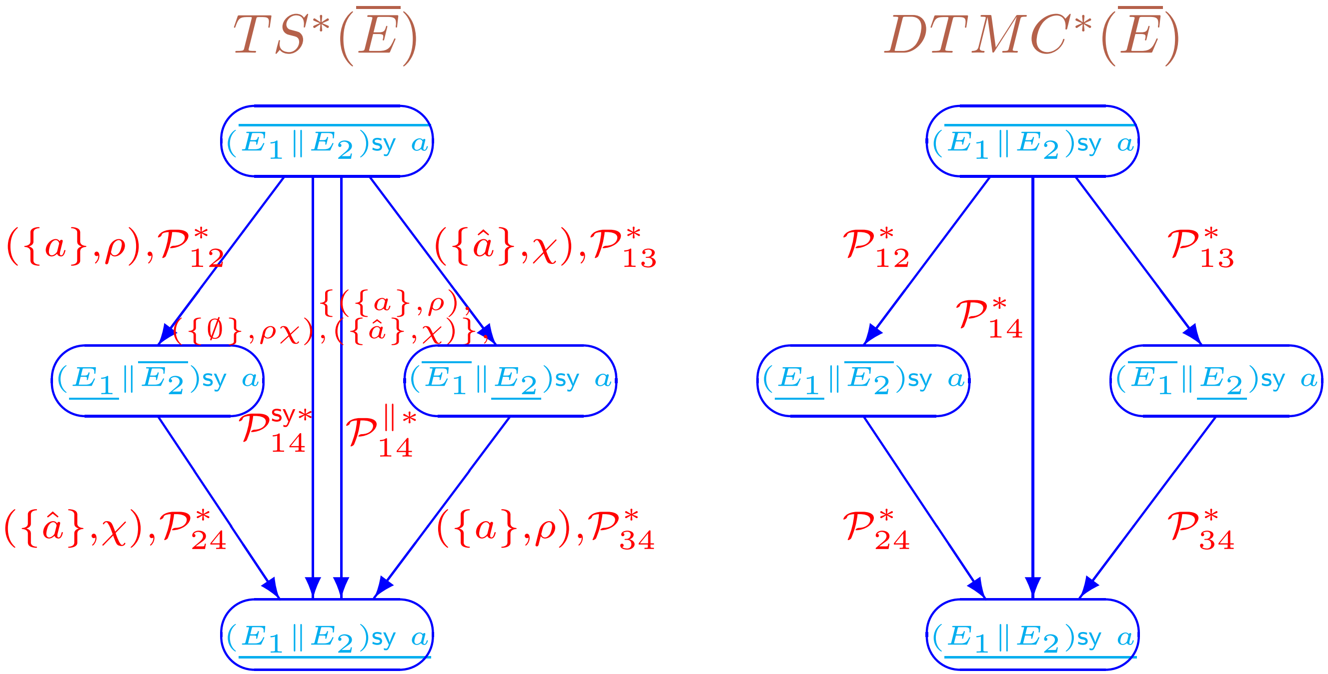
$$TS^*(\overline{E}) \simeq RG^*(Box_{dts}(\overline{E})).$$

Proposition 2 *For any static expression* E

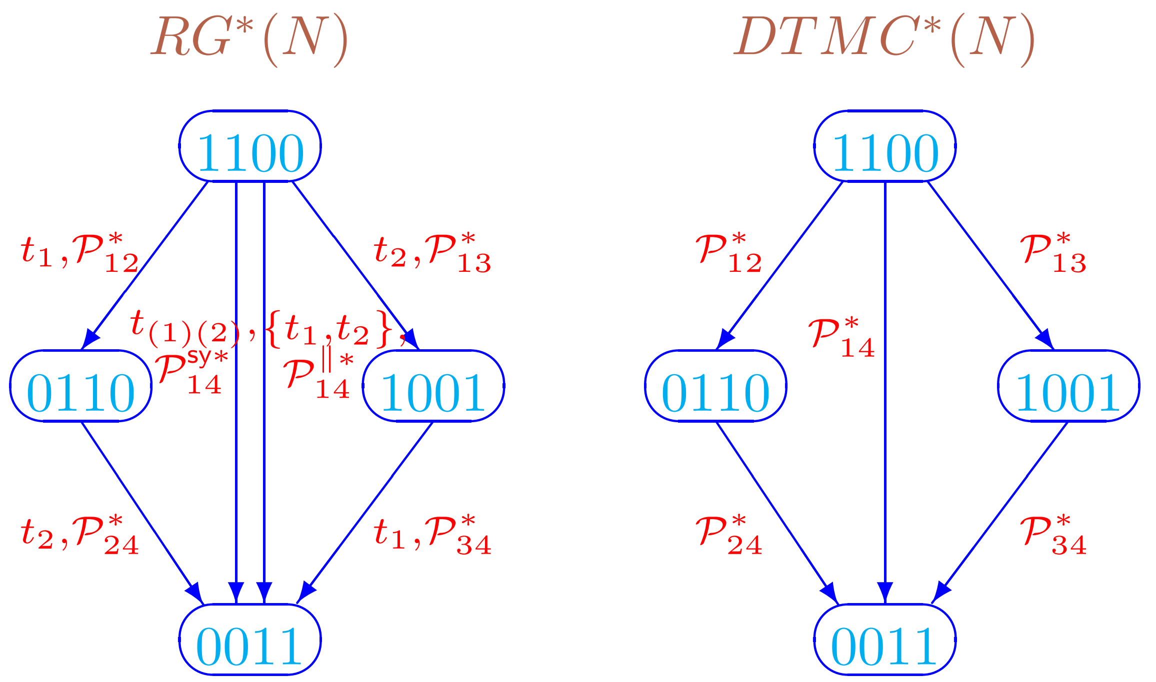
$$DTMC^*(\overline{E}) \simeq DTMC^*(Box_{dts}(\overline{E})).$$



The reachability graph and the underlying DTMC without empty loops of N in Figure BOXIT



The transition system and the underlying DTMC without empty loops of $\overline{E}$ in Figure EXPR



The reachability graph and the underlying DTMC without empty loops of N in Figure BOX

The normalization factor $\mathcal{N}^* = \frac{1}{\rho+\chi-2\rho^2\chi-2\rho\chi^2+2\rho^2\chi^2}$.

$$\mathcal{P}^*_{12} = \frac{\mathcal{P}_{12}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho(1-\chi)(1-\rho\chi)$$
$$\mathcal{P}^*_{13} = \frac{\mathcal{P}_{13}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\chi(1-\rho)(1-\rho\chi)$$
$$\mathcal{P}^{\mathrm{sy}*}_{14} = \frac{\mathcal{P}^{\mathrm{sy}}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(1-\rho)(1-\chi)$$
$$\mathcal{P}^{\|*}_{14} = \frac{\mathcal{P}^{\|}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(1-\rho\chi)$$
$$\mathcal{P}^*_{24} = \frac{\mathcal{P}_{24}}{1-\mathcal{P}_{22}} = 1$$
$$\mathcal{P}^*_{34} = \frac{\mathcal{P}_{34}}{1-\mathcal{P}_{33}} = 1$$
$$\mathcal{P}^*_{14} = \mathcal{P}^{\mathrm{sy}*}_{14} + \mathcal{P}^{\|*}_{14} = \frac{\mathcal{P}^{\mathrm{sy}}_{14}+\mathcal{P}^{\|}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(2-\rho-\chi)$$

The case $\rho = \chi = \frac{1}{2}$:

$$\mathcal{P}^*_{12} = \mathcal{P}^*_{13} = \mathcal{P}^{\|*}_{14} = \frac{3}{10},\ \mathcal{P}^{\mathrm{sy}*}_{14} = \frac{1}{10},\ \mathcal{P}^*_{24} = \mathcal{P}^*_{34} = 1,\ \mathcal{P}^*_{14} = \frac{2}{5}.$$

Stochastic trace equivalences

Let G be a dynamic expression, $s, \tilde{s} \in DR(G)$ and $s \stackrel{(\alpha,\rho)}{\rightarrow} \tilde{s}$. We write $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$, where $\mathcal{P} = pt^*((\alpha,\rho), s)$ is the *probability to execute the activity* (α,ρ) *in* s *after possible empty loops when only one-element steps are allowed*:

$$pt^*((\alpha,\rho), s) = \frac{PT^*(\{(\alpha,\rho)\}, s)}{\sum_{\{(\beta,\chi)\}\in Exec(s)} PT^*(\{(\beta,\chi)\}, s)}.$$

For $\Gamma \in I\!N_{fin}^{\mathcal{SL}}$, we consider $\mathcal{L}(\Gamma) \in I\!N_{fin}^{\mathcal{L}}$, i.e. (possibly empty) multisets of multiactions.

Definition 22 *An* interleaving stochastic trace *of a dynamic expression* G *is a pair* $(\sigma, pt^*(\sigma))$, *where* $\sigma = \alpha_1 \cdots \alpha_n \in \mathcal{L}^*$ *and*

$$pt^*(\sigma) = \sum_{\{(\alpha_1,\rho_1),\ldots,(\alpha_n,\rho_n) \mid [G]_\approx = s_0 \stackrel{(\alpha_1,\rho_1)}{\twoheadrightarrow} s_1 \stackrel{(\alpha_2,\rho_2)}{\twoheadrightarrow} \cdots \stackrel{(\alpha_n,\rho_n)}{\twoheadrightarrow} s_n\}} \prod_{i=1}^{n} pt^*((\alpha_i, \rho_i), s_{i-1}).$$

We denote a set of all interleaving stochastic traces *of a dynamic expression* G *by* $IntStochTraces(G)$.

G *and* G' *are* interleaving stochastic trace equivalent, $G \equiv_{is} G'$, *if*

$$IntStochTraces(G) = IntStochTraces(G').$$

Let $E = ((\{a\}, \frac{1}{2}) \| (\{\hat{a}\}, \frac{1}{2}))$ sy a.

$IntStochTraces(\overline{E}) = \{(\emptyset, \frac{1}{7}),\ (\{a\}, \frac{3}{7}),\ (\{\hat{a}\}, \frac{3}{7}),\ (\{a\}\{\hat{a}\}, \frac{3}{7}),\ (\{\hat{a}\}\{a\}, \frac{3}{7})\}$.

Definition 23 *A* step stochastic trace *of a dynamic expression* G *is a pair* $(\Sigma, PT^*(\Sigma))$, *where* $\Sigma = A_1 \cdots A_n \in (\mathbb{N}_{fin}^{\mathcal{L}} \setminus \{\emptyset\})^*$ *and*

$$PT^*(\Sigma) = \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\approx = s_0 \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i\ (1 \le i \le n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}).$$

We denote a set of all step stochastic traces *of a dynamic expression* G *by* $StepStochTraces(G)$.

G *and* G' *are* step stochastic trace equivalent, $G \equiv_{ss} G'$, *if*

$$StepStochTraces(G) = StepStochTraces(G').$$

Let $E = ((\{a\}, \frac{1}{2}) \| (\{\hat{a}\}, \frac{1}{2}))$ sy a.

$StepStochTraces(\overline{E}) = \{(\{\emptyset\}, \frac{1}{10}),\ (\{\{a\}\}, \frac{3}{10}),\ (\{\{\hat{a}\}\}, \frac{3}{10}),\ (\{\{a\}\}\{\{\hat{a}\}\}, \frac{3}{10}),$
$(\{\{\hat{a}\}\}\{\{a\}\}, \frac{3}{10}),\ (\{\{\hat{a}\}, \{a\}\}, \frac{3}{10})\}$.

Stochastic bisimulation equivalences

Let G be a dynamic expression and $\mathcal{H} \subseteq DR(G)$. For $s \in DR(G)$ and $A \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}$ we write $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = PM_A^*(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via non-empty steps with the multiaction part A after possible empty loops*:

$$PM_A^*(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s},\ \mathcal{L}(\Gamma) = A\}} PT^*(\Gamma, s).$$

We write $s \stackrel{A}{\twoheadrightarrow} \mathcal{H}$ if $\exists \mathcal{P}\ s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{H}$.

We write $s \twoheadrightarrow_{\mathcal{P}} \mathcal{H}$ if $\exists A\ s \stackrel{A}{\twoheadrightarrow} \mathcal{H}$, where $\mathcal{P} = PM^*(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via any non-empty steps after possible empty loops*:

$$PM^*(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}\}} PT^*(\Gamma, s).$$

We write $s \stackrel{\alpha}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = pm_\alpha^*(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via steps with the multiaction part $\{\alpha\}$ after possible empty loops when only one-element steps are allowed*:

$$pm_\alpha^*(s, \mathcal{H}) = \sum_{\{(\alpha,\rho) \mid \exists \tilde{s} \in \mathcal{H}\ s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}\}} pt^*((\alpha, \rho), s).$$

We write $s \stackrel{\alpha}{\twoheadrightarrow} \mathcal{H}$ if $\exists \mathcal{P}\ s \stackrel{\alpha}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{H}$.

Definition 24 *Let G and G' be dynamic expressions. An* equivalence *relation* $\mathcal{R} \subseteq (DR(G) \cup DR(G'))^2$ *is a* $\star$-stochastic bisimulation *between* G *and* G', $\star \in$ {interleaving, step}, $\mathcal{R} : G \underline{\leftrightarrow}_{\star s} G'$, $\star \in \{i, s\}$, *if:*

1. $([G]_\approx, [G']_\approx) \in \mathcal{R}$.
2. $(s_1, s_2) \in \mathcal{R} \Rightarrow \forall \mathcal{H} \in (DR(G) \cup DR(G'))/_\mathcal{R}$
   - $\forall x \in \mathcal{L}$ *and* $\hookrightarrow = \rightharpoonup$, *if* $\star = i$;
   - $\forall x \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}$ *and* $\hookrightarrow = \twoheadrightarrow$, *if* $\star = s$;

$$s_1 \stackrel{x}{\hookrightarrow}_\mathcal{P} \mathcal{H} \Leftrightarrow s_2 \stackrel{x}{\hookrightarrow}_\mathcal{P} \mathcal{H}.$$

Two dynamic expressions G *and* G' *are* $\star$-stochastic bisimulation equivalent, $\star \in$ {interleaving, step}, $G \underline{\leftrightarrow}_{\star s} G'$, *if* $\exists \mathcal{R} : G \underline{\leftrightarrow}_{\star s} G'$, $\star \in \{i, s\}$.

$\mathcal{R}_{\star s}(G, G') = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{\star s} G'\}$, $\star \in \{i, s\}$, is the *union of all* $\star$*-stochastic bisimulations* between G and G', $\star \in$ {*interleaving, step*}.

Proposition 3 *Let G and G' be dynamic expressions and* $G \underline{\leftrightarrow}_{\star s} G'$, $\star \in \{i, s\}$. *Then* $\mathcal{R}_{\star s}(G, G')$ *is the largest* $\star$*-stochastic bisimulation between* G *and* G', $\star \in$ {interleaving, step}.

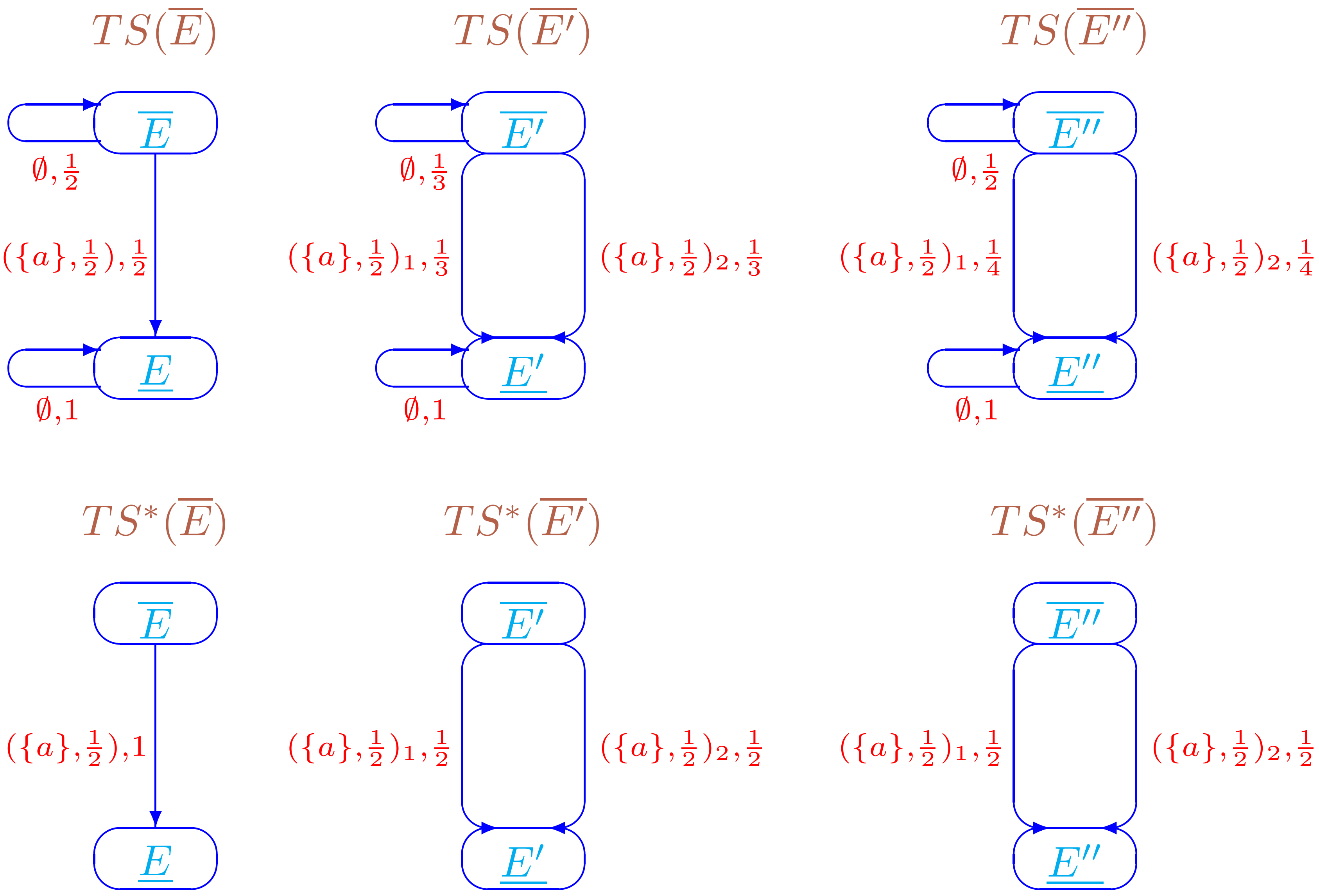
Stochastic isomorphism

Let G be a dynamic expression, $s, \tilde{s} \in DR(G)$ and $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \{\tilde{s}\}$. We write $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$.

Definition 25 *Let G, G' be dynamic expressions. A mapping $\beta : DR(G) \rightarrow DR(G')$ is a* stochastic isomorphism *between G and G', $\beta : G =_{sto} G'$, if*

1. *β is a bijection s.t. $\beta([G]_{\approx}) = [G']_{\approx}$;*
2. $\forall s, \tilde{s} \in DR(G)\ \forall A \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}\ s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s} \Leftrightarrow \beta(s) \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \beta(\tilde{s})$.

G and G' are stochastically isomorphic, *$G =_{sto} G'$, if $\exists \beta : G =_{sto} G'$.*



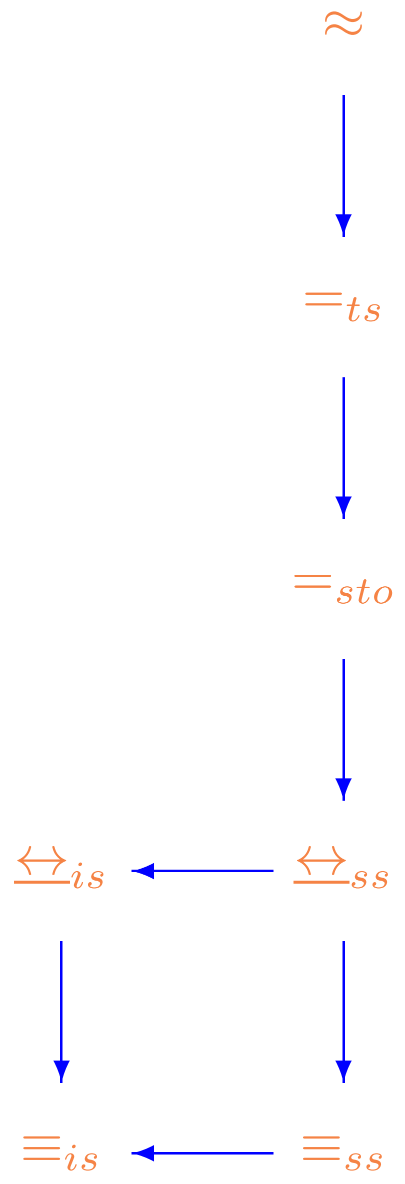
Properties of the stochastic isomorphism based on transition systems with empty loops

Let $E = (\{a\}, \frac{1}{2})$, $E' = (\{a\}, \frac{1}{2})_1[](\{a\}, \frac{1}{2})_2$, $E'' = (\{a\}, \frac{1}{3})_1[](\{a\}, \frac{1}{3})_2$.

The (one-element) multisets of activities which label the transitions of $TS^*(\overline{E})$, $TS^*(\overline{E'})$, $TS^*(\overline{E''})$, and non-empty ones of $TS(\overline{E})$, $TS(\overline{E'})$, $TS(\overline{E''})$, have the same multiaction part $\{\{a\}\}$.

- $\overline{E} =_{sto} \overline{E'} =_{sto} \overline{E''}$, since the probability of the only one non-empty transition in $TS^*(\overline{E})$ is 1, the probability of both non-empty transitions in $TS^*(\overline{E'})$ and $TS^*(\overline{E''})$ is $\frac{1}{2}$, and $1 = \frac{1}{2} + \frac{1}{2}$.
- $\overline{E}$ is not equivalent to $\overline{E'}$ w.r.t. the stronger version of stochastic isomorphism, since the probability of the only one non-empty transition in $TS(\overline{E})$ is $\frac{1}{2}$, whereas the probability of both non-empty transitions in $TS(\overline{E'})$ is $\frac{1}{3}$, and $\frac{1}{2} \neq \frac{2}{3} = \frac{1}{3} + \frac{1}{3}$.
- $\overline{E'}$ is not equivalent to $\overline{E''}$ w.r.t. the stronger version of stochastic isomorphism, since the probability of both non-empty transitions in $TS(\overline{E'})$ is $\frac{1}{3}$, whereas the probability of both non-empty transitions in $TS(\overline{E''})$ is $\frac{1}{4}$, and $\frac{1}{3} + \frac{1}{3} = \frac{2}{3} \neq \frac{1}{2} = \frac{1}{4} + \frac{1}{4}$.
- $\overline{E}$ is equivalent to $\overline{E''}$ w.r.t. the stronger version of stochastic isomorphism, since the probability of the only one non-empty transition in $TS(\overline{E})$ is $\frac{1}{2}$, the probability of both non-empty transitions in $TS(\overline{E''})$ is $\frac{1}{4}$, and $\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$.

Interrelations of the stochastic equivalences



Interrelations of the stochastic equivalences

Proposition 4 *Let* $\star \in \{i, s\}$. *For dynamic expressions* G *and* G':

1. $G \underline{\leftrightarrow}_{\star s} G' \;\Rightarrow\; G \equiv_{\star s} G'$;
2. $G =_{ts\ast} G' \;\Leftrightarrow\; G =_{ts} G'$.

Theorem 3 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, =, \approx\}$ *and* $\star, \star\star \in \{\_, is, ss, sto, ts\}$. *For dynamic expressions* G *and* G'

$$G \leftrightarrow_{\star} G' \;\Rightarrow\; G \twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star} G'$$

iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star}$.

Validity of the implications

- The implications $\leftrightarrow_{ss} \to \leftrightarrow_{is},\ \leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$ are valid, since single activities are one-element multisets.

- The implications $\underline{\leftrightarrow}_{\star s} \to \equiv_{\star s},\ \star \in \{i, s\}$, are valid by the proposition above.

- The implication $=_{sto} \to \underline{\leftrightarrow}_{ss}$ is proved as follows. Let $\beta : G =_{sto} G'$. Then $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$, where $\mathcal{R} = \{(s, \beta(s)) \mid s \in DR(G)\}$.

- The implication $=_{ts} \to =_{sto}$ is valid, since stochastic isomorphism is that of transition systems without empty loops up to merging of transitions with labels having identical multiaction parts.

- The implication $\approx \to =_{ts}$ is valid, since the transition system of a dynamic formula is defined based on its structural equivalence class.

Absence of the additional nontrivial arrows

(a) Let $E=(\{a\},\frac{1}{2})\|(\{b\},\frac{1}{2})$ and $E'=((\{a\},\frac{1}{2});(\{b\},\frac{1}{2}))[]((\{b\},\frac{1}{2});(\{a\},\frac{1}{2}))$. Then $\overline{E}\underline{\leftrightarrow}_{is}\overline{E'}$, but $\overline{E}\not\equiv_{ss}\overline{E'}$, since only in $TS^*(\overline{E'})$ multiactions $\{a\}$ and $\{b\}$ cannot be executed concurrently.

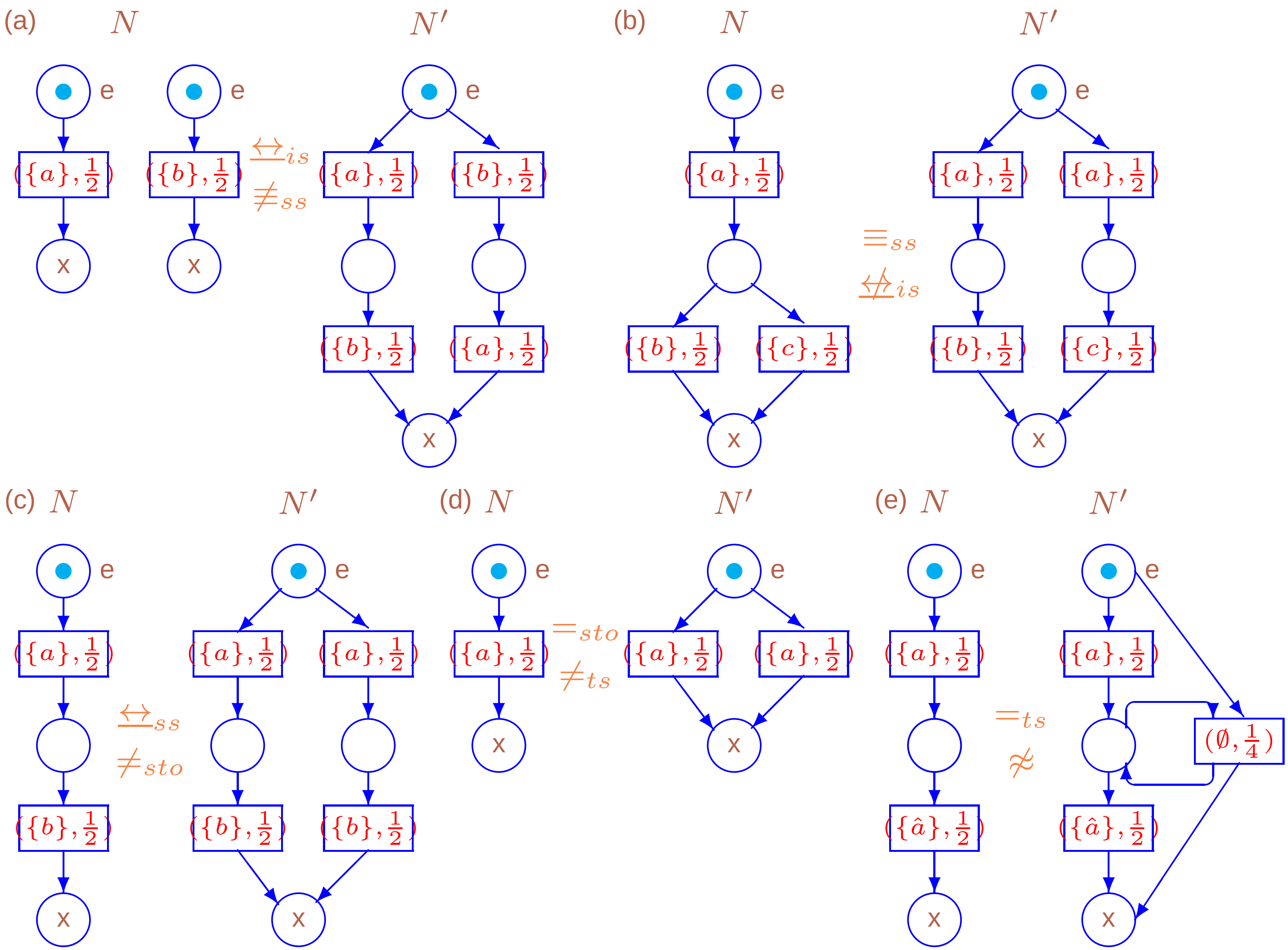
(b) Let $E=(\{a\},\frac{1}{2});((\{b\},\frac{1}{2})[](\{c\},\frac{1}{2}))$ and $E'=((\{a\},\frac{1}{2});(\{b\},\frac{1}{2}))[]((\{a\},\frac{1}{2});(\{c\},\frac{1}{2}))$. Then $\overline{E}\equiv_{ss}\overline{E'}$, but $\overline{E}\not\underline{\leftrightarrow}_{is}\overline{E'}$, since only in $TS^*(\overline{E'})$ a multiaction $\{a\}$ can be executed so that no multiaction $\{b\}$ can occur afterwards.

(c) Let $E=(\{a\},\frac{1}{2});(\{b\},\frac{1}{2})$ and $E'=(\{a\},\frac{1}{2});(\{b\},\frac{1}{2})[](\{a\},\frac{1}{2});(\{b\},\frac{1}{2})$. Then $\overline{E}\underline{\leftrightarrow}_{ss}\overline{E'}$, but $\overline{E}\neq_{sto}\overline{E'}$, since $TS^*(\overline{E'})$ has more states than $TS^*(\overline{E})$.

(d) Let $E=(\{a\},\frac{1}{2})$ and $E'=(\{a\},\frac{1}{2})_1[](\{a\},\frac{1}{2})_2$. Then $\overline{E}=_{sto}\overline{E'}$, but $\overline{E}\neq_{ts}\overline{E'}$, since only $TS(\overline{E'})$ has two transitions.

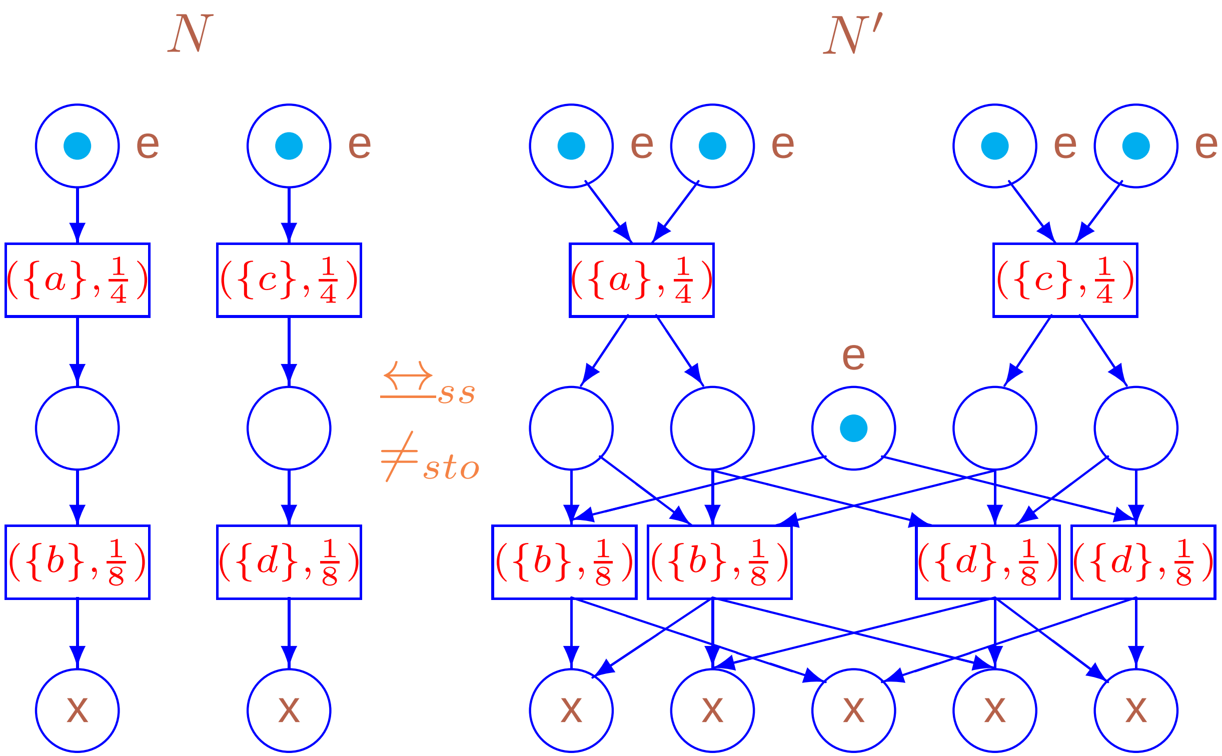
(e) Let $E=(\{a\},\frac{1}{2});(\{\hat{a}\},\frac{1}{2})$ and $E'=((\{a\},\frac{1}{2});(\{\hat{a}\},\frac{1}{2}))$ sy a. Then $\overline{E}=_{ts}\overline{E'}$, but $\overline{E}\not\approx\overline{E'}$, since $\overline{E}$ and $\overline{E'}$ cannot be reached each from other by applying inaction rules.

In the figure below $N=Box_{dts}(\overline{E})$ and $N'=Box_{dts}(\overline{E'})$ for each picture (a)–(e).



Dts-boxes of the dynamic expressions from equivalence examples of the theorem above

Reduction modulo equivalences



Reduction of a dts-box up to $\underline{\leftrightarrow}_{ss}$

Let $E = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))\|((\{c\}, \frac{1}{2}); (\{d\}, \frac{1}{2}))$ and $E' = ((((\{a, x\}, \frac{1}{2}); (((\{b, y_1\}, \frac{1}{2})[] (\{b, y_2\}, \frac{1}{2})))\|((\{a, \hat{x}\}, \frac{1}{2}); (((\{b, \widehat{y_2}, y_2'\}, \frac{1}{2})[](\{d, v_1\}, \frac{1}{2})))\|((\{c, z\}, \frac{1}{2}); (((\{b, \widehat{y_2'}\}, \frac{1}{2})[] (\{d, \widehat{v_1}, v_1'\}, \frac{1}{2})))\|((\{c, \hat{z}\}, \frac{1}{2}); (((\{d, \widehat{v_1'}\}, \frac{1}{2})[](\{d, v_2\}, \frac{1}{2})))\|(((\{b, \widehat{y_1}\}, \frac{1}{4})[](\{d, \widehat{v_2}\}, \frac{1}{4}))$ sy x sy y_1 sy y_2 sy y_2' sy z sy v_1 sy v_1' sy v_2 rs x rs y_1 rs y_2 rs y_2' rs z rs v_1 rs v_1' rs v_2.

We have $\overline{E} \underline{\leftrightarrow}_{ss} \overline{E'}$, but $\overline{E} \not= _{sto} \overline{E'}$, since $TS^*(\overline{E'})$ has more states than $TS^*(\overline{E})$.

Thus, E is a reduction of E' w.r.t. $\underline{\leftrightarrow}_{ss}$.

For $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'}),\ N$ is a reduction of N' w.r.t. the net version of $\underline{\leftrightarrow}_{ss}$.

An *autobisimulation* is a bisimulation between an expression and itself.

For a dynamic expression G and a step stochastic autobisimulation $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G$, let $\mathcal{K} \in DR(G)/_{\mathcal{R}}$ and $s_1, s_2 \in \mathcal{K}$.

We have $\forall \widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}}\ \forall A \in I\!N^{\mathcal{L}}_{fin} \setminus \{\emptyset\}\ s_1 \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}} \ \Leftrightarrow\ s_2 \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

The equality is valid for all $s_1, s_2 \in \mathcal{K}$, hence, we can rewrite it as $\mathcal{K} \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM^*_A(\mathcal{K}, \widetilde{\mathcal{K}}) = PM^*_A(s_1, \widetilde{\mathcal{K}}) = PM^*_A(s_2, \widetilde{\mathcal{K}})$.

We write $\mathcal{K} \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}$ if $\exists \mathcal{P}\ \mathcal{K} \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$ and $\mathcal{K} \twoheadrightarrow \widetilde{\mathcal{K}}$ if $\exists A\ \mathcal{K} \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}$.

The similar arguments: we write $\mathcal{K} \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM^*(\mathcal{K}, \widetilde{\mathcal{K}}) = PM^*(s_1, \widetilde{\mathcal{K}}) = PM^*(s_2, \widetilde{\mathcal{K}})$.

$\mathcal{R}_{ss}(G) = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{ss} G\}$ is the *largest step stochastic autobisimulation* on G.

Definition 26 *The* quotient (by $\underline{\leftrightarrow}_{ss}$) (labeled probabilistic) transition system without empty loops *of a dynamic expression* G *is* $TS^*_{\underline{\leftrightarrow}_{ss}}(G) = (S_{\underline{\leftrightarrow}_{ss}}, L_{\underline{\leftrightarrow}_{ss}}, \mathcal{T}_{\underline{\leftrightarrow}_{ss}}, s_{\underline{\leftrightarrow}_{ss}})$*, where*

- $S_{\underline{\leftrightarrow}_{ss}} = DR(G)/_{\mathcal{R}_{ss}(G)}$;
- $L_{\underline{\leftrightarrow}_{ss}} \subseteq (\mathbb{N}^{\mathcal{L}}_{fin} \setminus \{\emptyset\}) \times (0; 1]$;
- $\mathcal{T}_{\underline{\leftrightarrow}_{ss}} = \{(\mathcal{K}, (A, PM^*_A(\mathcal{K}, \widetilde{\mathcal{K}})), \widetilde{\mathcal{K}}) \mid \mathcal{K}, \widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}_{ss}(G)},\ \mathcal{K} \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}$;
- $s_{\underline{\leftrightarrow}_{ss}} = [[G]_{\approx}]_{\mathcal{R}_{ss}(G)}$.

The transition $(\mathcal{K}, (A, \mathcal{P}), \widetilde{\mathcal{K}}) \in \mathcal{T}_{\underline{\leftrightarrow}_{ss}}$ will be written as $\mathcal{K} \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

For $E \in RegStatExpr$, let $TS^*_{\underline{\leftrightarrow}_{ss}}(E) = TS^*_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Definition 27 *The* quotient (by $\underline{\leftrightarrow}_{ss}$) underlying DTMC without empty loops *of a dynamic expression* G, $DTMC^*_{\underline{\leftrightarrow}_{ss}}(G)$, *has the state space* $DR(G)/_{\mathcal{R}_{ss}(G)}$, *the initial state* $[[G]_{\approx}]_{\mathcal{R}_{ss}(G)}$ *and the transitions* $\mathcal{K} \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, *where* $\mathcal{P} = PM^*(\mathcal{K}, \widetilde{\mathcal{K}})$.

For $E \in RegStatExpr$, let $DTMC^*_{\underline{\leftrightarrow}_{ss}}(E) = DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Logical characterization

Logic iPML

Definition 28 $\top$ *is the truth,* $\alpha \in \mathcal{L}$, $\mathcal{P} \in (0;1]$. *A* formula *of* $iPML$*:*

$$\Phi ::= \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \nabla_\alpha \mid \langle\alpha\rangle_{\mathcal{P}}\Phi$$

iPML is the set of *all formulas of the logic* $iPML$.

Definition 29 *Let* G *be a dynamic expression and* $s \in DR(G)$*. The* satisfaction relation $\models_G \subseteq DR(G) \times \mathbf{iPML}$*:*

1. $s \models_G \top$ *— always;*
2. $s \models_G \neg\Phi$*, if* $s \not\models_G \Phi$*;*
3. $s \models_G \Phi\wedge\Psi$*, if* $s \models_G \Phi$ *and* $s \models_G \Psi$*;*
4. $s \models_G \nabla_\alpha$*, if not* $s \stackrel{\alpha}{\rightarrow} DR(G)$*;*
5. $s \models_G \langle\alpha\rangle_{\mathcal{P}}\Phi$*, if* $\exists\mathcal{H} \subseteq DR(G)\ s \stackrel{\alpha}{\rightarrow}_{\mathcal{Q}} \mathcal{H},\ \mathcal{Q} \geq \mathcal{P}$ *and* $\forall\tilde{s} \in \mathcal{H}\ \tilde{s} \models_G \Phi$.

$\langle\alpha\rangle\Phi = \exists\mathcal{P}\ \langle\alpha\rangle_{\mathcal{P}}\Phi$. $\langle\alpha\rangle_{\mathcal{Q}}\Phi$ implies $\langle\alpha\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

We write $G \models_G \Phi$, if $[G]_\approx \models_G \Phi$.

Definition 30 *G and G' are logically equivalent in $iPML$, $G =_{iPML} G'$, if*
$\forall \Phi \in \mathbf{iPML}\ G \models_G \Phi \ \Leftrightarrow\ G' \models_{G'} \Phi$.

Let G be a dynamic expression and $s \in DR(G),\ \alpha \in \mathcal{L}$.

The set of states reached from s by execution of α, the *image set*, is

$Image(s, \alpha) = \{\tilde{s} \mid \exists \{(\alpha, \rho)\} \in Exec(s)\ s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}\}$.

A dynamic expression G is an *image-finite* one, if $\forall s \in DR(G)\ \forall \alpha \in \mathcal{L}\ |Image(s, \alpha)| < \infty$.

Theorem 4 *For image-finite dynamic expressions G and G'*

$$G \underline{\leftrightarrow}_{is} G' \ \Leftrightarrow\ G =_{iPML} G'.$$

Let $E = (\{a\}, \frac{1}{2}); ((\{b\}, \frac{1}{2})[](\{c\}, \frac{1}{2}))$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{a\}, \frac{1}{2}); (\{c\}, \frac{1}{2}))$.
Then $\overline{E} \neq_{iPML} \overline{E'}$, because for $\Phi = \langle \{a\} \rangle_1 \langle \{b\} \rangle_{\frac{1}{2}} \top$ we have $\overline{E} \models_{\overline{E}} \Phi$, but $\overline{E'} \not\models_{\overline{E'}} \Phi$, since in $TS^*(\overline{E'})$ a multiaction $\{a\}$ can be executed so that no multiaction $\{b\}$ can occur afterwards.

Logic sPML

Definition 31 $\top$ *is the truth,* $A \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}$, $\mathcal{P} \in (0;1]$.

A formula of $sPML$*:*

$$\Phi ::= \ \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \nabla_A \mid \langle A\rangle_{\mathcal{P}}\Phi$$

sPML is the set of *all formulas of the logic* $sPML$.

Definition 32 *Let* G *be a dynamic expression and* $s \in DR(G)$. *The satisfaction relation* $\models_G \subseteq DR(G) \times \mathbf{sPML}$*:*

1. $s \models_G \top$ — *always;*
2. $s \models_G \neg\Phi$, *if* $s \not\models_G \Phi$*;*
3. $s \models_G \Phi\wedge\Psi$, *if* $s \models_G \Phi$ *and* $s \models_G \Psi$*;*
4. $s \models_G \nabla_A$, *if not* $s \stackrel{A}{\twoheadrightarrow} DR(G)$*;*
5. $s \models_G \langle A\rangle_{\mathcal{P}}\Phi$, *if* $\exists\mathcal{H} \subseteq DR(G)\ s \stackrel{A}{\twoheadrightarrow}_{\mathcal{Q}} \mathcal{H},\ \mathcal{Q} \geq \mathcal{P}$ *and* $\forall\tilde{s} \in \mathcal{H}\ \tilde{s} \models_G \Phi$.

$\langle A\rangle\Phi = \exists\mathcal{P}\ \langle A\rangle_{\mathcal{P}}\Phi$. $\langle A\rangle_{\mathcal{Q}}\Phi$ implies $\langle A\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

We write $G \models_G \Phi$, if $[G]_\approx \models_G \Phi$.

Definition 33 *G and G' are logically equivalent in $sPML$, $G =_{sPML} G'$, if $\forall\Phi \in \mathbf{sPML}\ G \models_G \Phi \ \Leftrightarrow\ G' \models_{G'} \Phi$.*

Let G be a dynamic expression and $s \in DR(G),\ A \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}$.

The set of states reached from s by execution of A, the *image set*, is
$Image(s, A) = \{\tilde{s} \mid \exists\Gamma \in Exec(s)\ \mathcal{L}(\Gamma) = A,\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}\}$.

A dynamic expression G is an *image-finite* one, if
$\forall s \in DR(G)\ \forall A \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}\ |Image(s, A)| < \infty$.

Theorem 5 *For image-finite dynamic expressions G and G'*

$$G \underline{\leftrightarrow}_{ss} G' \ \Leftrightarrow\ G =_{sPML} G'.$$

Let $E = (\{a\}, \frac{1}{2}) \| (\{b\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{b\}, \frac{1}{2}); (\{a\}, \frac{1}{2}))$. Then $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$ but $\overline{E} \neq_{sPML} \overline{E'}$, because for $\Phi = \langle\{a, b\}\rangle_{\frac{1}{3}} \top$ we have $\overline{E} \models_{\overline{E}} \Phi$, but $\overline{E'} \not\models_{\overline{E'}} \Phi$, since in $TS^*(\overline{E'})$ multiactions $\{a\}$ and $\{b\}$ cannot be executed concurrently.

Stationary behaviour

Theoretical background

The elements $\mathcal{P}^*_{ij}$ $(1 \leq i, j \leq n = |DR(G)|)$ of *(one-step) transition probability matrix (TPM)* $\mathbf{P}^*$ for $DTMC^*(G)$:

$$\mathcal{P}^*_{ij} = \begin{cases} PM^*(s_i, s_j), & s_i \twoheadrightarrow s_j; \\ 0, & \text{otherwise}. \end{cases}$$

The *transient (k-step, $k \in I\!N$) probability mass function (PMF)* $\psi^*[k] = (\psi^*_1[k], \ldots, \psi^*_n[k])$ for $DTMC^*(G)$ is calculated as

$$\psi^*[k] = \psi^*[0](\mathbf{P}^*)^k,$$

where $\psi^*[0] = (\psi_1^*[0], \ldots, \psi_n^*[0])$ is the *initial PMF*:

$$\psi_i^*[0] = \begin{cases} 1, & s_i = [G]_{\approx}; \\ 0, & \text{otherwise}. \end{cases}$$

We have $\psi^*[k+1] = \psi^*[k]\mathbf{P}^*,\ k \in I\!N$.

The *steady-state PMF* $\psi^* = (\psi_1^*, \ldots, \psi_n^*)$ for $DTMC^*(G)$ is a solution of

$$\begin{cases} \psi^*(\mathbf{P}^* - \mathbf{I}) = \mathbf{0} \\ \psi^*\mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{I}$ is the identity matrix of order n, $\mathbf{0}$ is a vector of n values $0,$ $\mathbf{1}$ is that of n values 1.

When $DTMC^*(G)$ has the single steady state, $\psi^* = \lim_{k\to\infty} \psi^*[k]$.

For $s \in DR(G)$ with $s = s_i \ (1 \leq i \leq n)$ we define $\psi^*[k](s) = \psi_i^*[k] \ (k \in I\!N)$ and $\psi^*(s) = \psi_i^*$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G), \ S, \widetilde{S} \subseteq DR(G)$.

The following performance indices (measures) are based on the steady-state PMF.

- The *average recurrence (return) time in the state* s (the number of discrete time units or steps required for this) is $\frac{1}{\psi^*(s)}$.
- The *fraction of residence time in the state* s is $\psi^*(s)$.
- The *fraction of residence time in the set of states* $S \subseteq DR(G)$ or the *probability of the event determined by a condition that is true for all states from* S is $\sum_{s \in S} \psi^*(s)$.
- The *relative fraction of residence time in the set of states* S *w.r.t. that in* $\widetilde{S}$ is $\frac{\sum_{s \in S} \psi^*(s)}{\sum_{\tilde{s} \in \widetilde{S}} \psi^*(\tilde{s})}$.
- The *steady-state probability to perform a step with a multiset of activities* Δ is $\sum_{s \in DR(G)} \psi^*(s) \sum_{\{\Gamma \mid \Delta \subseteq \Gamma\}} PT^*(\Gamma, s)$.
- The *probability of the event determined by a reward function* r *on the states* is $\sum_{s \in DR(G)} \psi^*(s) r(s)$, where $\forall s \in DR(G) \ 0 \leq r(s) \leq 1$.

Theorem 6 *Let* G *be a dynamic expression and* EL *be its empty loops abstraction vector. The steady-state PMFs* ψ *for* $DTMC(G)$ *and* ψ^* *for* $DTMC^*(G)$ *are related as:* $\forall s \in DR(G)$

$$\psi(s) = \frac{\psi^*(s)EL(s)}{\sum_{\tilde{s}\in DR(G)} \psi^*(\tilde{s})EL(\tilde{s})}.$$

Steady state and equivalences

Proposition 5 *Let* G, G' *be dynamic expressions with* $\mathcal{R} : G\underline{\leftrightarrow}_{ss}G'$ *and* ψ^* *be the steady-state PMF for* $DTMC^*(G)$, ψ'^* *be the steady-state PMF for* $DTMC^*(G')$*. Then* $\forall\mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

$$\sum_{s\in\mathcal{H}\cap DR(G)} \psi^*(s) = \sum_{s'\in\mathcal{H}\cap DR(G')} \psi'^*(s').$$

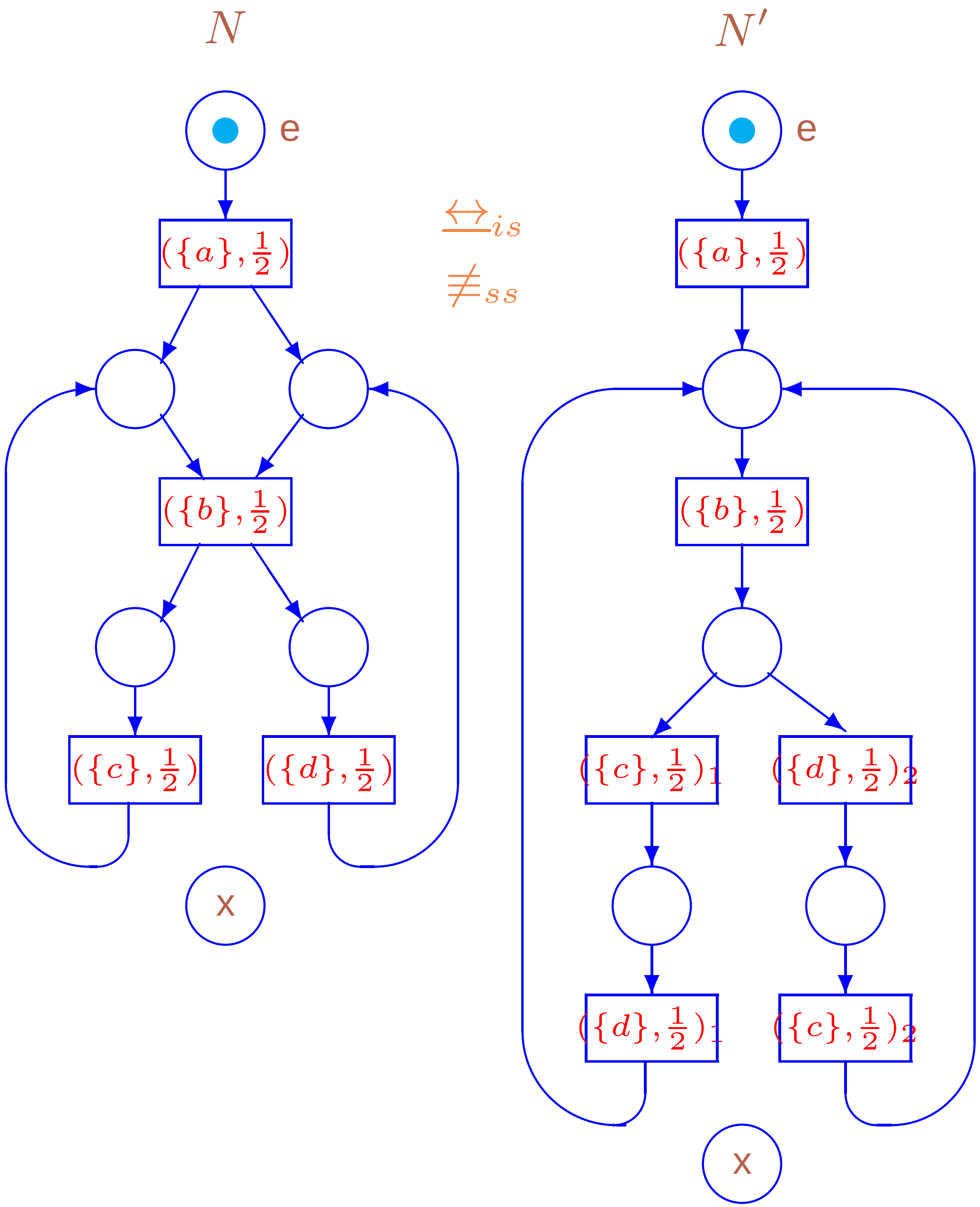
The result of the proposition above is valid if we replace steady-state probabilities with transient ones.

Let G be a dynamic expression. The transient PMF $\psi^*_{\underline{\leftrightarrow}_{ss}}[k]$ ($k \in \mathbb{N}$) and the steady-state PMF $\psi^*_{\underline{\leftrightarrow}_{ss}}$ for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(G)$ are defined like the corresponding notions $\psi^*[k]$ and ψ^* for $DTMC^*(G)$.

By the proposition above: $\forall\mathcal{K} \in DR(G)/_{\mathcal{R}_{ss}(G)}$

$$\psi^*_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}) = \sum_{s\in\mathcal{K}} \psi^*(s).$$

$\mathsf{Stop} = (\{c\}, \frac{1}{2})$ rs c is the process that performs empty loops with probability 1 and never terminates.



$\underline{\leftrightarrow}_{is}$ does not guarantee a coincidence of steady-state probabilities to enter into an equivalence class

Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2}) \| (\{d\}, \frac{1}{2}))) * \mathsf{Stop}]$ and
$E' = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))) * \mathsf{Stop}]$.
We have $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$.

$DR(\overline{E})$ consists of

$$s_1 = [[\overline{(\{a\}, \frac{1}{2})} * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2}) \| (\{d\}, \frac{1}{2}))) * \mathsf{Stop}]]_{\approx},$$
$$s_2 = [[(\{a\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{1}{2})}; ((\{c\}, \frac{1}{2}) \| (\{d\}, \frac{1}{2}))) * \mathsf{Stop}]]_{\approx},$$
$$s_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (\overline{(\{c\}, \frac{1}{2})} \| \overline{(\{d\}, \frac{1}{2})})) * \mathsf{Stop}]]_{\approx},$$
$$s_4 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (\underline{(\{c\}, \frac{1}{2})} \| \overline{(\{d\}, \frac{1}{2})})) * \mathsf{Stop}]]_{\approx},$$
$$s_5 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (\overline{(\{c\}, \frac{1}{2})} \| \underline{(\{d\}, \frac{1}{2})})) * \mathsf{Stop}]]_{\approx}.$$

$DR(\overline{E'})$ consists of

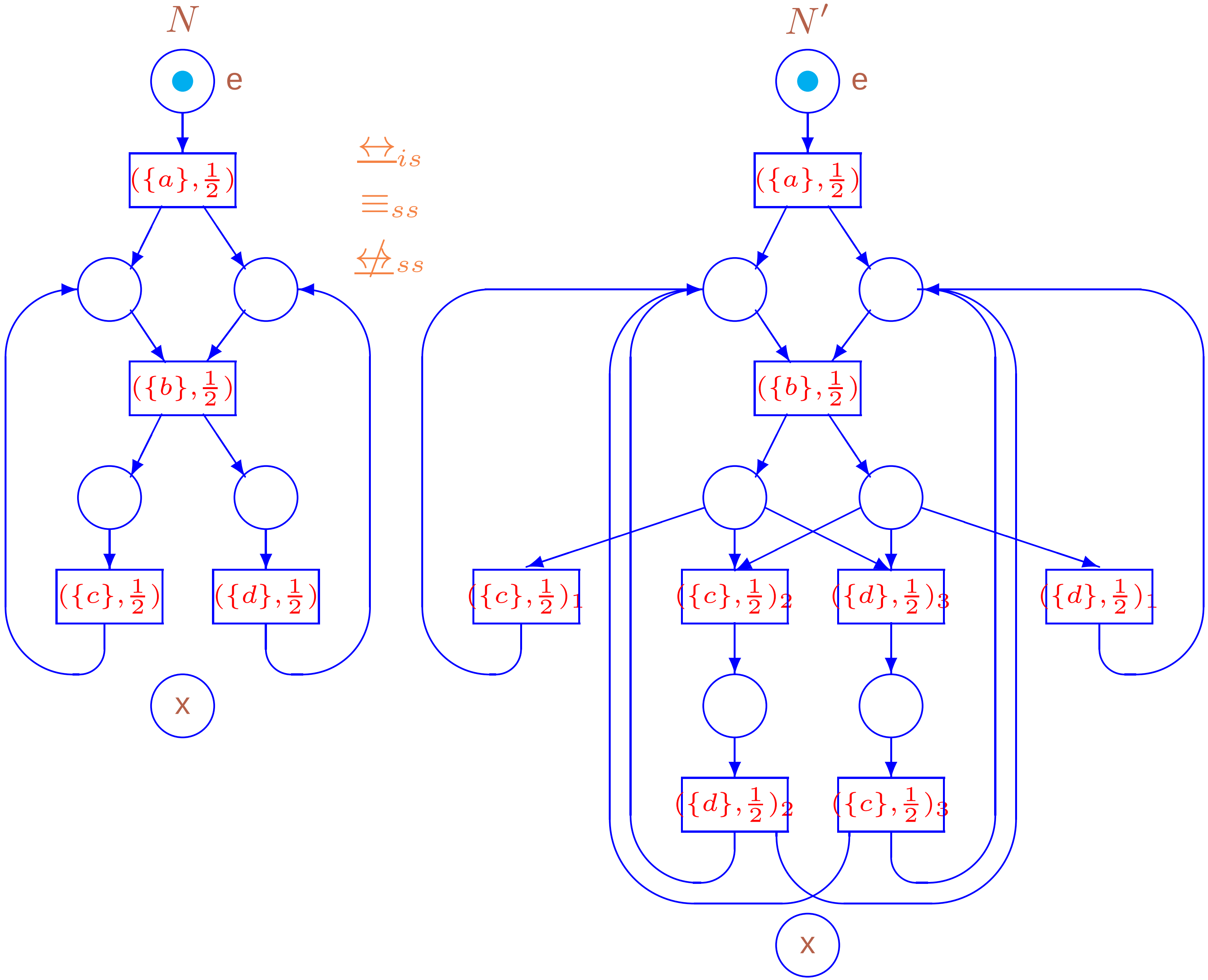
$$s'_1 = [[\overline{(\{a\}, \frac{1}{2})} * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))) * \mathsf{Stop}]]_{\approx},$$
$$s'_2 = [[(\{a\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{1}{2})}; (((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))) * \mathsf{Stop}]]_{\approx},$$
$$s'_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); \overline{(((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))}) * \mathsf{Stop}]]_{\approx},$$
$$s'_4 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1; \overline{(\{d\}, \frac{1}{2})_1}) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))) * \mathsf{Stop}]]_{\approx},$$
$$s'_5 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; \overline{(\{c\}, \frac{1}{2})_2}))) * \mathsf{Stop}]]_{\approx}.$$

The steady-state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right),\ \psi'^* = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right).$$

Consider $\mathcal{H} = \{s_3, s_3'\}$. We have $\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{3}{8}$, whereas $\sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} \psi'^*(s') = \psi'^*(s_3') = \frac{1}{3}$. Thus, $\underline{\leftrightarrow}_{is}$ does not guarantee a coincidence of steady-state probabilities to enter into an equivalence class.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.



The intersection of $\underline{\leftrightarrow}_{is}$ and $\equiv_{ss}$ does not guarantee a coincidence of steady-state probabilities to enter into an equivalence class

Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2}) \| (\{d\}, \frac{1}{2}))) * \mathsf{Stop}]$ and $E' = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1)) [] (((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2) [] ((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]$.

We have $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$ and $\overline{E} \equiv_{ss} \overline{E'}$.

$DR(\overline{E})$ is as in the previous example.

$DR(\overline{E'})$ consists of

$$s'_1 = [[\overline{(\{a\},\tfrac{1}{2})} * ((\{b\},\tfrac{1}{2}); ((((\{c\},\tfrac{1}{2})_1 \| (\{d\},\tfrac{1}{2})_1)) [] ((((\{c\},\tfrac{1}{2})_2; (\{d\},\tfrac{1}{2})_2) [] ((\{d\},\tfrac{1}{2})_3; (\{c\},\tfrac{1}{2})_3))))) * \mathsf{Stop}]]_{\approx},$$

$$s'_2 = [[(\{a\},\tfrac{1}{2}) * (\overline{(\{b\},\tfrac{1}{2})}; ((((\{c\},\tfrac{1}{2})_1 \| (\{d\},\tfrac{1}{2})_1)) [] ((((\{c\},\tfrac{1}{2})_2; (\{d\},\tfrac{1}{2})_2) [] ((\{d\},\tfrac{1}{2})_3; (\{c\},\tfrac{1}{2})_3))))) * \mathsf{Stop}]]_{\approx},$$

$$s'_3 = [[(\{a\},\tfrac{1}{2}) * ((\{b\},\tfrac{1}{2}); (\overline{(((\{c\},\tfrac{1}{2})_1 \| (\{d\},\tfrac{1}{2})_1)) [] ((((\{c\},\tfrac{1}{2})_2; (\{d\},\tfrac{1}{2})_2) [] ((\{d\},\tfrac{1}{2})_3; (\{c\},\tfrac{1}{2})_3)))})) * \mathsf{Stop}]]_{\approx},$$

$$s'_4 = [[(\{a\},\tfrac{1}{2}) * ((\{b\},\tfrac{1}{2}); (((\underline{(\{c\},\tfrac{1}{2})_1} \| \overline{(\{d\},\tfrac{1}{2})_1})) [] ((((\{c\},\tfrac{1}{2})_2; (\{d\},\tfrac{1}{2})_2) [] ((\{d\},\tfrac{1}{2})_3; (\{c\},\tfrac{1}{2})_3))))) * \mathsf{Stop}]]_{\approx},$$

$$s'_5 = [[(\{a\},\tfrac{1}{2}) * ((\{b\},\tfrac{1}{2}); (((\overline{(\{c\},\tfrac{1}{2})_1} \| \underline{(\{d\},\tfrac{1}{2})_1})) [] ((((\{c\},\tfrac{1}{2})_2; (\{d\},\tfrac{1}{2})_2) [] ((\{d\},\tfrac{1}{2})_3; (\{c\},\tfrac{1}{2})_3))))) * \mathsf{Stop}]]_{\approx},$$

$$s'_6 = [[(\{a\},\tfrac{1}{2}) * ((\{b\},\tfrac{1}{2}); ((((\{c\},\tfrac{1}{2})_1 \| (\{d\},\tfrac{1}{2})_1)) [] ((((\{c\},\tfrac{1}{2})_2; \overline{(\{d\},\tfrac{1}{2})_2}) [] ((\{d\},\tfrac{1}{2})_3; (\{c\},\tfrac{1}{2})_3))))) * \mathsf{Stop}]]_{\approx},$$

$$s'_7 = [[(\{a\},\tfrac{1}{2}) * ((\{b\},\tfrac{1}{2}); ((((\{c\},\tfrac{1}{2})_1 \| (\{d\},\tfrac{1}{2})_1)) [] ((((\{c\},\tfrac{1}{2})_2; (\{d\},\tfrac{1}{2})_2) [] ((\{d\},\tfrac{1}{2})_3; \overline{(\{c\},\tfrac{1}{2})_3}))))) * \mathsf{Stop}]]_{\approx}.$$

The steady-state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right),\ \psi'^* = \left(0, \frac{13}{38}, \frac{13}{38}, \frac{3}{38}, \frac{3}{38}, \frac{3}{38}, \frac{3}{38}\right).$$

Consider $\mathcal{H} = \{s_3, s_3'\}$. We have $\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{3}{8}$, whereas $\sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} \psi'^*(s') = \psi'^*(s_3') = \frac{13}{38}$. Thus, $\underline{\leftrightarrow}_{is}$ plus $\equiv_{ss}$ do not guarantee a coincidence of steady-state probabilities to enter into an equivalence class.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.

Definition 34 *A derived step trace of a dynamic expression* G *is* $\Sigma = A_1 \cdots A_n \in (\mathbb{N}_{fin}^{\mathcal{L}} \setminus \{\emptyset\})^*$, *where* $\exists s \in DR(G)\ s \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i) = A_i\ (1 \leq i \leq n)$.

The probability to execute the derived step trace Σ *in* s*:*

$$PT^*(\Sigma, s) = \sum_{\{\Gamma_1,\ldots,\Gamma_n | s=s_0 \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i\ (1\leq i\leq n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}).$$

Theorem 7 *Let* G, G' *be dynamic expressions with* $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ *and* ψ^* *be the steady-state PMF for* $DTMC^*(G)$, ψ'^* *be the steady-state PMF for* $DTMC^*(G')$ *and* Σ *be a derived step trace of* G *and* G'. *Then* $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

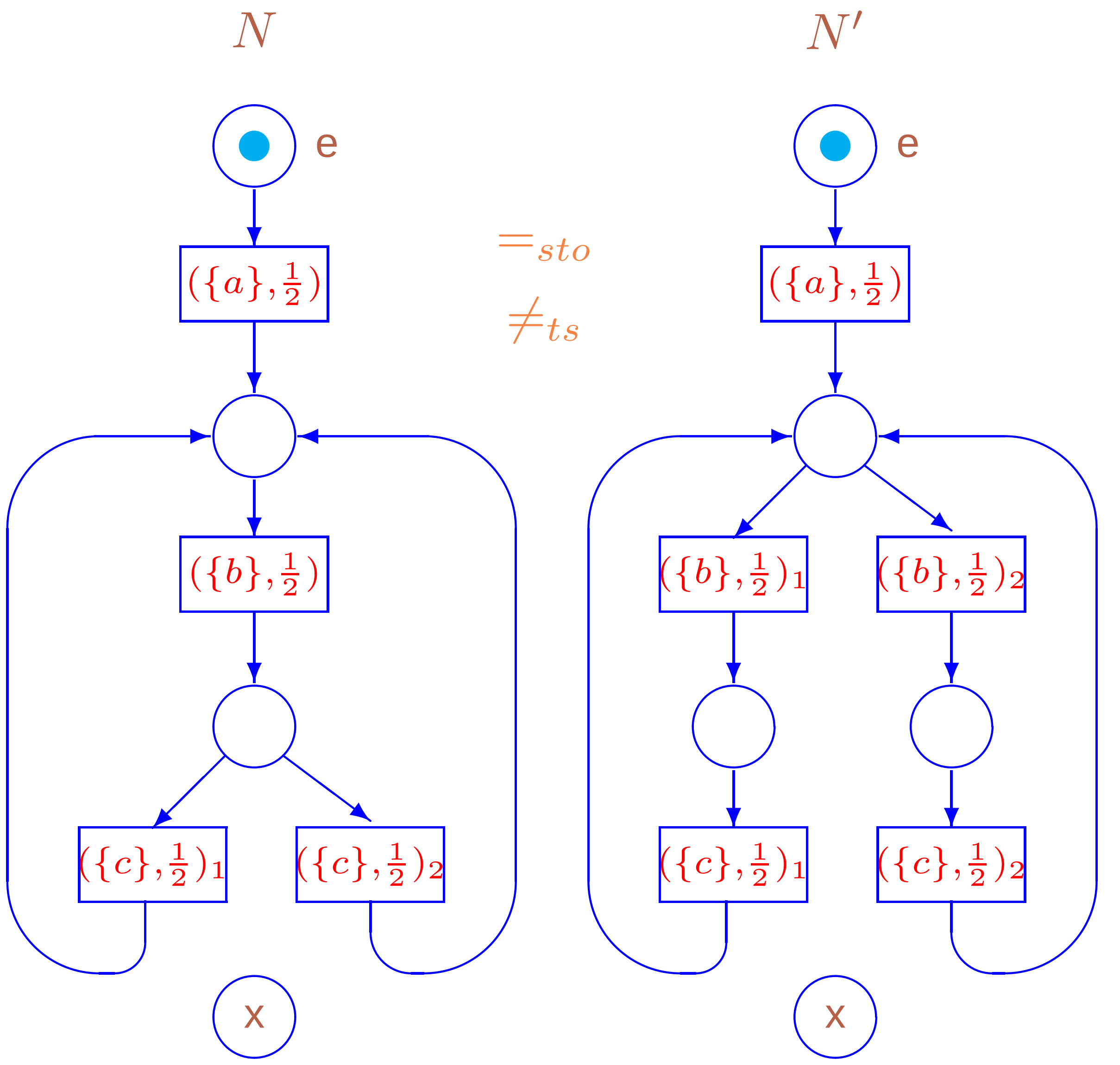
$$\sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) PT^*(\Sigma, s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \psi'^*(s') PT^*(\Sigma, s').$$

The result of the theorem above is valid if we replace steady-state probabilities with transient ones.

By the theorem above: $\forall\mathcal{K}\in DR(G)/_{\mathcal{R}_{ss}(G)}$

$$\psi^*_{\underline{\leftrightarrow}_{ss}}(\mathcal{K})PT^*(\Sigma,\mathcal{K})=\sum_{s\in\mathcal{K}}\psi^*(s)PT^*(\Sigma,s),$$

where $\forall s\in\mathcal{K}\ PT^*(\Sigma,\mathcal{K})=PT^*(\Sigma,s)$.



$\underline{\leftrightarrow}_{ss}$ preserves steady-state behaviour in the equivalence classes

Let $E=[(\{a\},\frac{1}{2})*((\{b\},\frac{1}{2});((\{c\},\frac{1}{2})_1[](\{c\},\frac{1}{2})_2))*\mathsf{Stop}]$ and
$E'=[(\{a\},\frac{1}{2})*(((\{b\},\frac{1}{2})_1;(\{c\},\frac{1}{2})_1)[]((\{b\},\frac{1}{2})_2;(\{c\},\frac{1}{2})_2))*\mathsf{Stop}]$.
We have $\overline{E}=_{sto}\overline{E'}$, hence, $\overline{E}\underline{\leftrightarrow}_{ss}\overline{E'}$.

$DR(\overline{E})$ consists of

$s_1=[[\overline{(\{a\},\frac{1}{2})}*((\{b\},\frac{1}{2});((\{c\},\frac{1}{2})_1[](\{c\},\frac{1}{2})_2))*\mathsf{Stop}]]_\approx$,
$s_2=[[(\{a\},\frac{1}{2})*(\overline{(\{b\},\frac{1}{2})};((\{c\},\frac{1}{2})_1[](\{c\},\frac{1}{2})_2))*\mathsf{Stop}]]_\approx$,
$s_3=[[(\{a\},\frac{1}{2})*((\{b\},\frac{1}{2});\overline{((\{c\},\frac{1}{2})_1[](\{c\},\frac{1}{2})_2)})*\mathsf{Stop}]]_\approx$.

$DR(\overline{E'})$ consists of

$s_1'=[[\overline{(\{a\},\frac{1}{2})}*(((\{b\},\frac{1}{2})_1;(\{c\},\frac{1}{2})_1)[]((\{b\},\frac{1}{2})_2;(\{c\},\frac{1}{2})_2))*\mathsf{Stop}]]_\approx$,
$s_2'=[[(\{a\},\frac{1}{2})*\overline{(((\{b\},\frac{1}{2})_1;(\{c\},\frac{1}{2})_1)[]((\{b\},\frac{1}{2})_2;(\{c\},\frac{1}{2})_2))}*\mathsf{Stop}]]_\approx$,
$s_3'=[[(\{a\},\frac{1}{2})*(((\{b\},\frac{1}{2})_1;\overline{(\{c\},\frac{1}{2})_1})[]((\{b\},\frac{1}{2})_2;(\{c\},\frac{1}{2})_2))*\mathsf{Stop}]]_\approx$,
$s_4'=[[(\{a\},\frac{1}{2})*(((\{b\},\frac{1}{2})_1;(\{c\},\frac{1}{2})_1)[]((\{b\},\frac{1}{2})_2;\overline{(\{c\},\frac{1}{2})_2}))*\mathsf{Stop}]]_\approx$.

The steady-state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{1}{2}, \frac{1}{2}\right),\ \psi'^* = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Consider $\mathcal{H} = \{s_3, s_3', s_4'\}$. The steady-state probabilities for $\mathcal{H}$ coincide:
$\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \psi'^*(s_3') + \psi'^*(s_4') = \sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} \psi'^*(s')$.

Let $\Sigma = \{\{c\}\}$. The steady-state probabilities to enter into the equivalence class $\mathcal{H}$ and start the derived step trace Σ from it coincide:
$\psi^*(s_3)(PT^*(\{(\{c\}, \frac{1}{2})_1\}, s_3) + PT^*(\{(\{c\}, \frac{1}{2})_2\}, s_3)) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2} = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 =$
$\psi'^*(s_3')PT^*(\{(\{c\}, \frac{1}{2})_1\}, s_3') + \psi'^*(s_4')PT^*(\{(\{c\}, \frac{1}{2})_2\}, s_4')$.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.

Simplification of performance analysis

The method of performance analysis simplification.

1. The system under investigation is specified by a static expression of $dtsPBC$.
2. The transition system without empty loops of the expression is constructed.
3. After examining this transition system for self-similarity and symmetry, a step stochastic autobisimulation equivalence for the expression is determined.
4. The quotient underlying DTMC without empty loops of the expression is constructed from the quotient transition system without empty loops.
5. The steady-state probabilities and performance indices based on this DTMC are calculated.

$$E \rightarrow TS^*(\overline{E}) \rightarrow TS^*_{\underline{\leftrightarrow}^*_{ss}}(\overline{E}) \rightarrow DTMC^*_{\underline{\leftrightarrow}^*_{ss}}(\overline{E}) \rightarrow \psi^*_{\underline{\leftrightarrow}^*_{ss}} \rightarrow \text{Performance}$$

Equivalence-based simplification of performance evaluation

The limitation of the method: the expressions with underlying DTMCs containing one closed communication class of states, which is ergodic, to ensure uniqueness of the stationary distribution.

If a DTMC contains several closed communication classes of states that are all ergodic: several stationary distributions may exist, depending on the initial PMF.

The general steady-state probabilities are then calculated as the sum of the stationary probabilities of all the ergodic classes of states, weighted by the probabilities to enter these classes, starting from the initial state and passing through transient states.

The underlying DTMC of each process expression has one initial PMF (that at the time moment 0): the stationary distribution is unique.

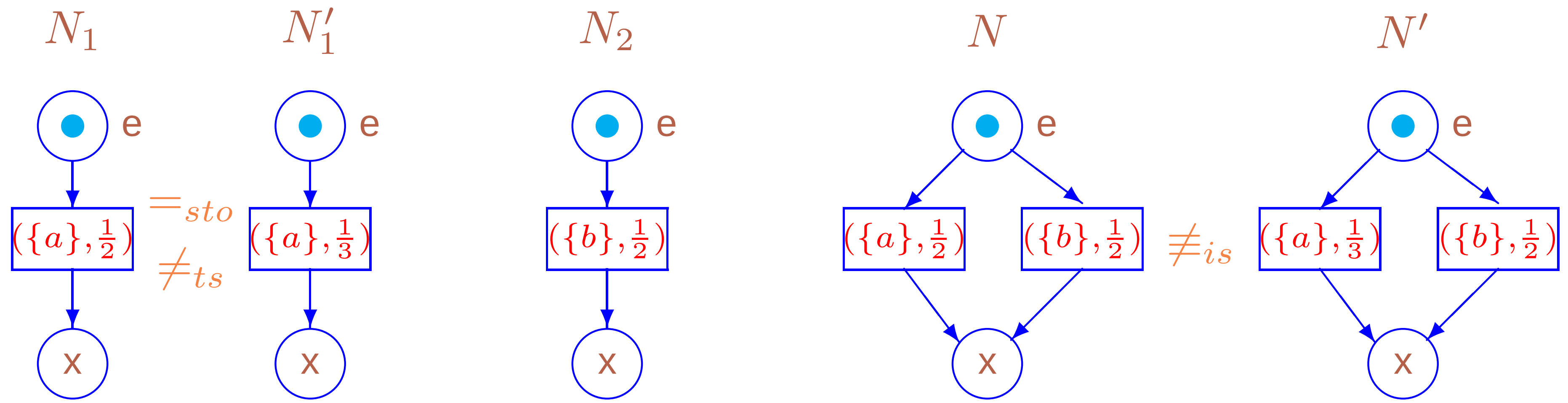
It is worth applying the method to the systems with similar subprocesses.

Preservation by algebraic operations

Definition 35 *Let $\leftrightarrow$ be an equivalence of dynamic expressions. Static expressions E and E' are* equivalent w.r.t. $\leftrightarrow$, $E \leftrightarrow E'$, *if* $\overline{E} \leftrightarrow \overline{E'}$.

Proposition 6 *Let* $\star \in \{is, ss\}$, $\star\star \in \{sto, ts\}$. *The equivalences* $\equiv_\star$, $\underline{\leftrightarrow}_\star$, $=_{\star\star}$ *are not preserved by algebraic operations.*

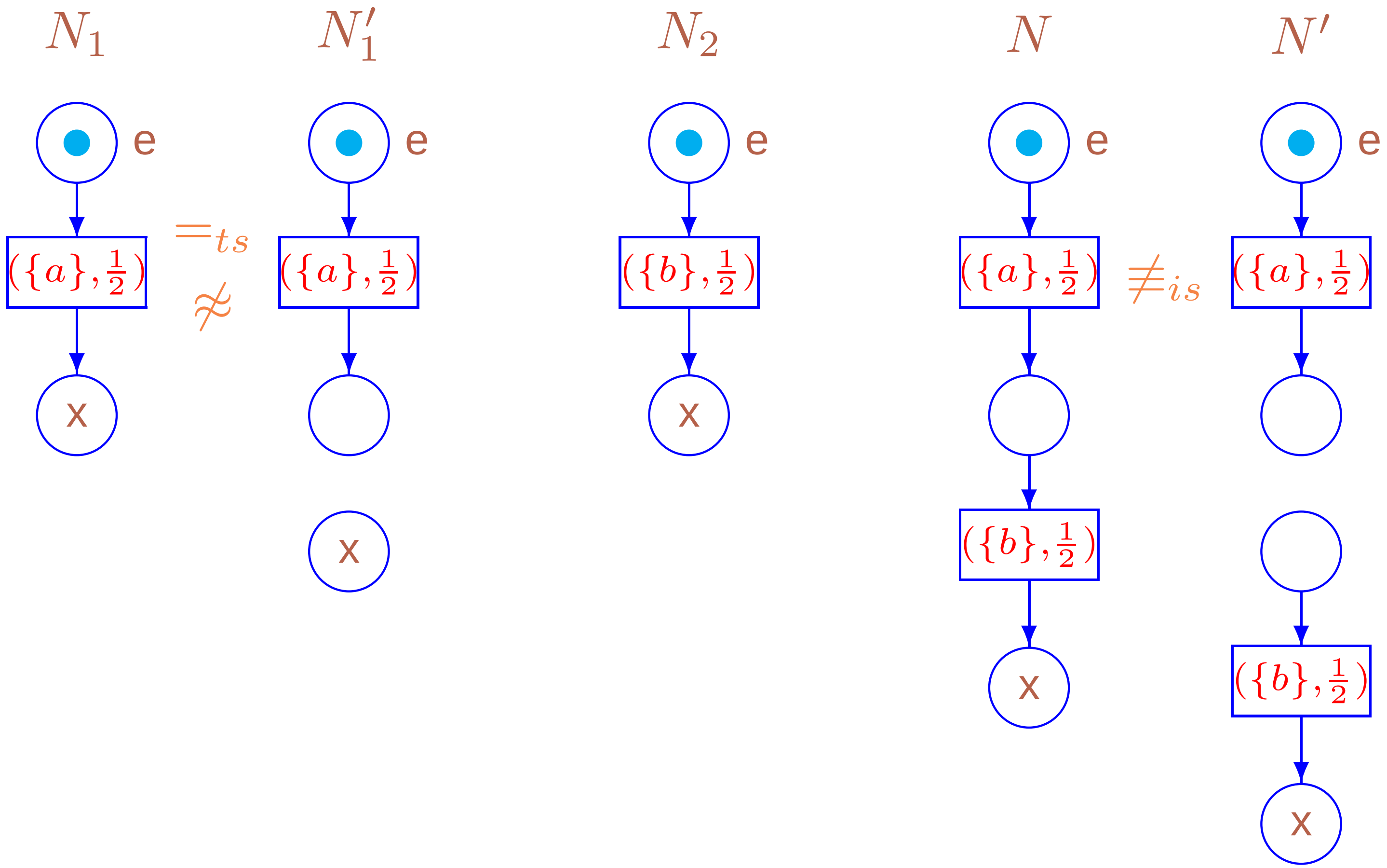
Proposition 7 *The equivalence $\approx$ is preserved by algebraic operations.*



SC1: The equivalences between $\equiv_{is}$ and $=_{sto}$ are not congruences

Let $E = (\{a\}, \frac{1}{2}),\ E' = (\{a\}, \frac{1}{3})$ and $F = (\{b\}, \frac{1}{2})$. We have $\overline{E} =_{sto} \overline{E'}$, since $TS^*(\overline{E})$ and $TS^*(\overline{E'})$ have the transitions with the multiaction part of labels $\{a\}$ and probability 1. $\overline{E[]F} \not\equiv_{is} \overline{E'[]F}$, since only in $TS^*(\overline{E'[]F})$ the probabilities of the transitions with the multiaction parts of labels $\{a\}$ and $\{b\}$ are different ($\frac{1}{3}$ and $\frac{2}{3}$, respectively). Thus, no equivalence between $\equiv_{is}$ and $=_{sto}$ is a congruence.

In the figure above $N_1 = Box_{dts}(\overline{E}),\ N_1' = Box_{dts}(\overline{E'}),\ N_2 = Box_{dts}(\overline{F})$ and $N = Box_{dts}(\overline{E[]F}),\ N' = Box_{dts}(\overline{E'[]F})$.



SC2: The equivalences between $\equiv_{is}$ and $=_{ts}$ are not congruences

Let $E = (\{a\}, \frac{1}{2})$, $E' = (\{a\}, \frac{1}{2})$; Stop and $F = (\{b\}, \frac{1}{2})$. We have $\overline{E} =_{ts} \overline{E'}$, since both $TS(\overline{E})$ and $TS(\overline{E'})$ have the transitions with the multiaction part of labels $\{a\}$ and probability $\frac{1}{2}$. $\overline{E; F} \not\equiv_{is} \overline{E'; F}$, since only in $TS^*(\overline{E'; F})$ no other transition can fire after the transition with the multiaction part of label $\{a\}$. Thus, no equivalence between $\equiv_{is}$ and $=_{ts}$ is a congruence. In the figure above $N_1 = Box_{dts}(\overline{E})$, $N_1' = Box_{dts}(\overline{E'})$, $N_2 = Box_{dts}(\overline{F})$ and $N = Box_{dts}(\overline{E; F})$, $N' = Box_{dts}(\overline{E'; F})$.

For an analogue of $=_{ts}$ to be a congruence, we have to equip transition systems with two extra transitions skip and redo as in [MVC02].

The equivalences between $\equiv_{is}$ and $=_{sto}$ defined on the basis of the enriched transition systems will still be non-congruences by Example SC1.

Rules for skip and redo: skipping and redoing all executions.

Let $E \in RegStatExpr$.

Rules for skip and redo

$$\textbf{Sk}\ \overline{E} \stackrel{\mathsf{skip}}{\rightarrow} \underline{E} \qquad \textbf{Rd}\ \underline{E} \stackrel{\mathsf{redo}}{\rightarrow} \overline{E}$$

Definition 36 *Let E be a static expression and $TS(\overline{E}) = (S, L, \mathcal{T}, s)$. The* (labeled probabilistic) sr-transition system *of $\overline{E}$ is a quadruple $TS_{sr}(\overline{E}) = (S_{sr}, L_{sr}, \mathcal{T}_{sr}, s_{sr})$:*

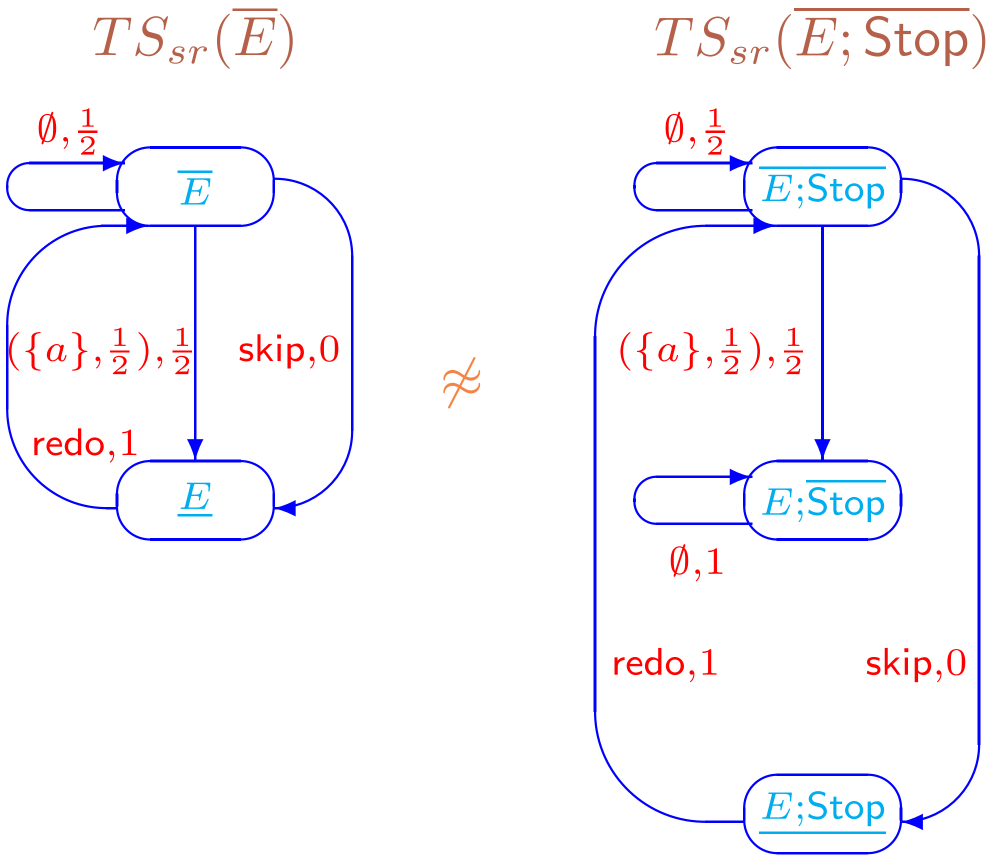
- $S_{sr} = S \cup \{[\underline{E}]_\approx\}$;
- $L_{sr} \subseteq (\mathbb{N}_{fin}^{\mathcal{SL}} \times (0;1]) \cup \{(\mathsf{skip}, 0), (\mathsf{redo}, 1)\}$;
- $\mathcal{T}_{sr} = \mathcal{T} \setminus \{([\underline{E}]_\approx, (\emptyset, 1), [\underline{E}]_\approx)\} \cup \{([\overline{E}]_\approx, (\mathsf{skip}, 0), [\underline{E}]_\approx), ([\underline{E}]_\approx, (\mathsf{redo}, 1), [\overline{E}]_\approx)\}$;
- $s_{sr} = s$.

Definition 37 *Let* E, E' *be static expressions and* $TS_{sr}(\overline{E}) = (S_{sr}, L_{sr}, \mathcal{T}_{sr}, s_{sr})$, $TS_{sr}(\overline{E'}) = (S'_{sr}, L'_{sr}, \mathcal{T}'_{sr}, s'_{sr})$ *be their* sr*-transition systems. A mapping* $\beta : S_{sr} \to S'_{sr}$ *is an* isomorphism *between* $TS_{sr}(\overline{E})$ *and* $TS_{sr}(\overline{E'})$, $\beta : TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$, *if*

1. β *is a bijection s.t.* $\beta(s_{sr}) = s'_{sr}$ *and* $\beta([\underline{E}]_{\approx}) = [\underline{E'}]_{\approx}$;
2. $\forall s, \tilde{s} \in S_{sr}\ \forall \Gamma\ s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s}\ \Leftrightarrow\ \beta(s) \xrightarrow{\Gamma}_{\mathcal{P}} \beta(\tilde{s})$.

Two sr*-transition systems* $TS_{sr}(\overline{E})$ *and* $TS_{sr}(\overline{E'})$ *are* isomorphic, $TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$, *if* $\exists \beta : TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$.

For $E \in RegStatExpr$, let $TS_{sr}(E) = TS_{sr}(\overline{E})$.



TSSR: The sr-transition systems of $\overline{E}$ and $\overline{E;\text{Stop}}$ for $E = (\{a\}, \frac{1}{2})$

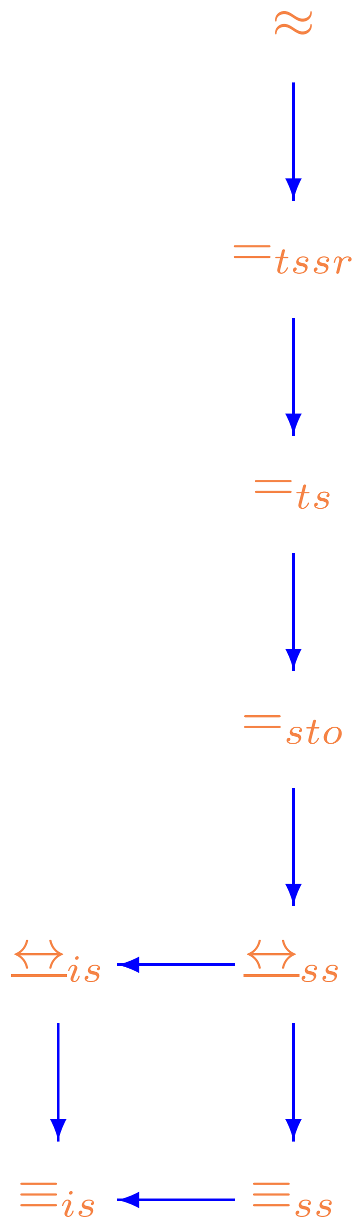
Let $E = (\{a\}, \frac{1}{2})$. In the figure above the transition systems $TS_{sr}(\overline{E})$ and $TS_{sr}(\overline{E;\text{Stop}})$ are presented.

In the latter sr-transition system the final state can be reached by the transition $(\text{skip}, 0)$ only from the initial state .

Definition 38 $\overline{E}$ *and* $\overline{E'}$ *are* equivalent w.r.t. sr-transition systems, $\overline{E} =_{tssr} \overline{E'}$, *if* $TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$.

sr*-transition systems without empty loops* can be defined and the equivalence $=_{tssr*}$ based on them.

The coincidence of $=_{tssr}$ and $=_{tssr*}$ can be proved as for $=_{ts}$ and $=_{ts*}$.



Interrelations of the stochastic equivalences and the new congruence

Theorem 8 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, =, \approx\}$ *and* $\star, \star\star \in \{\_, is, ss, sto, ts, tssr\}$*. For dynamic expressions* G *and* G'

$$G \leftrightarrow_{\star} G' \;\Rightarrow\; G \twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star} G'$$

iff in the graph in figure above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star}$.

Validity of the implications

- The implication $=_{tssr} \to =_{ts}$ is valid, since sr-transition systems have more states and transitions than usual ones.
- The implication $\approx \to =_{tssr}$ is valid, since the sr-transition system of a dynamic formula is defined based on its structural equivalence class.

Absence of the additional nontrivial arrows

- Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2}); \mathsf{Stop}$. We have $\overline{E} =_{ts} \overline{E'}$ (see example with Figure SC2). On the other hand, $\overline{E} \neq_{tssr} \overline{E'}$, since only in $TS_{sr}(\overline{E'})$ after the transition with multiaction part of label $\{a\}$ we do not reach the final state (see Figure TSSR).

- Let $E = (\{a\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2}))$ sy a. Then $\overline{E} =_{tssr} \overline{E'}$, since $\overline{E} =_{ts} \overline{E'}$ by the last example from the equivalence interrelations theorem, and the final states of both $TS_{sr}(\overline{E'})$ and $TS_{sr}(\overline{E'})$ are reachable from the others with “normal” transitions (not with skip only). On the other hand, $\overline{E} \not\approx \overline{E'}$.

Theorem 9 *Let* $a \in Act$ *and* $E, E', F \in RegStatExpr$. *If* $\overline{E} =_{tssr} \overline{E'}$ *then*

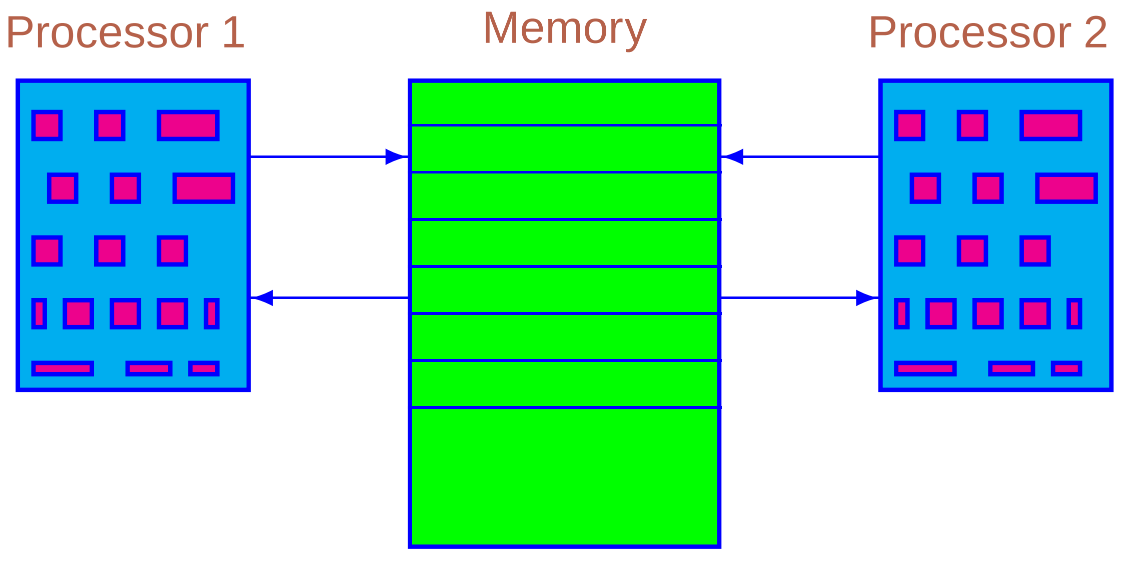
1. $\overline{E \circ F} =_{tssr} \overline{E' \circ F},\ \overline{F \circ E} =_{tssr} \overline{F \circ E'},\ \circ \in \{;, [], \|\}$;
2. $\overline{E[f]} =_{tssr} \overline{E'[f]}$;
3. $\overline{E \circ a} =_{tssr} \overline{E' \circ a},\ \circ \in \{\mathsf{rs}, \mathsf{sy}\}$;
4. $\overline{[E * F * K]} =_{tssr} \overline{[E' * F * K]},\ \overline{[F * E * K]} =_{tssr} \overline{[F * E' * K]},\ \overline{[F * K * E]} =_{tssr} \overline{[F * K * E']}$.

Case studies

Shared memory system

The standard system

A model of two processors accessing a common shared memory [MBCDF95]



The diagram of the shared memory system

After activation of the system (turning the computer on), two processors are active, and the common memory is available. Each processor can request an access to the memory.

When a processor starts an acquisition of the memory, another processor waits until the former one ends its operations, and the system returns to the state with both active processors and the available memory.

a corresponds to the system activation.

r_i $(1 \leq i \leq 2)$ represent the common memory request of processor i.

b_i and e_i correspond to the beginning and the end of the common memory access of processor i.

The other actions are used for communication purpose only.

The static expression of the first processor is

$E_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r_1\}, \frac{1}{2}); (\{b_1, y_1\}, \frac{1}{2}); (\{e_1, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$E_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r_2\}, \frac{1}{2}); (\{b_2, y_2\}, \frac{1}{2}); (\{e_2, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is

$E_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, \frac{1}{2}); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \frac{1}{2}); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the shared memory system with two processors is

$E = (E_1 \| E_2 \| E_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

Effect of synchronization

The synchronization of $(\{b_i, y_i\}, \frac{1}{2})$ and $(\{\widehat{y_i}\}, \frac{1}{2})$ produces $(\{b_i\}, \frac{1}{4})$ $(1 \leq i \leq 2)$.

The synchronization of $(\{e_i, z_i\}, \frac{1}{2})$ and $(\{\widehat{z_i}\}, \frac{1}{2})$ produces $(\{e_i\}, \frac{1}{4})$ $(1 \leq i \leq 2)$.

The synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2})$ and $(\{x_1\}, \frac{1}{2})$ produces $(\{a, \widehat{x_2}\}, \frac{1}{4})$,

Synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2})$ and $(\{x_2\}, \frac{1}{2})$ produces $(\{a, \widehat{x_1}\}, \frac{1}{4})$.

Synchronization of $(\{a, \widehat{x_2}\}, \frac{1}{4})$ and $(\{x_2\}, \frac{1}{2})$, as well as $(\{a, \widehat{x_1}\}, \frac{1}{4})$ and $(\{x_1\}, \frac{1}{2})$ produces $(\{a\}, \frac{1}{8})$.

$DR(\overline{E})$ consists of

$$s_1 = [([\overline{(\{x_1\},\tfrac{1}{2})} * ((\{r_1\},\tfrac{1}{2});(\{b_1,y_1\},\tfrac{1}{2});(\{e_1,z_1\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[\overline{(\{x_2\},\tfrac{1}{2})} * ((\{r_2\},\tfrac{1}{2});(\{b_2,y_2\},\tfrac{1}{2});(\{e_2,z_2\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[\overline{(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2})} * (((\{\widehat{y_1}\},\tfrac{1}{2});(\{\widehat{z_1}\},\tfrac{1}{2}))[]((\{\widehat{y_2}\},\tfrac{1}{2});(\{\widehat{z_2}\},\tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_2 = [([(\{x_1\},\tfrac{1}{2}) * (\overline{(\{r_1\},\tfrac{1}{2})};(\{b_1,y_1\},\tfrac{1}{2});(\{e_1,z_1\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * (\overline{(\{r_2\},\tfrac{1}{2})};(\{b_2,y_2\},\tfrac{1}{2});(\{e_2,z_2\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2}) * \overline{(((\{\widehat{y_1}\},\tfrac{1}{2});(\{\widehat{z_1}\},\tfrac{1}{2}))[]((\{\widehat{y_2}\},\tfrac{1}{2});(\{\widehat{z_2}\},\tfrac{1}{2})))} * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_3 = [([(\{x_1\},\tfrac{1}{2}) * ((\{r_1\},\tfrac{1}{2});\overline{(\{b_1,y_1\},\tfrac{1}{2})};(\{e_1,z_1\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * (\overline{(\{r_2\},\tfrac{1}{2})};(\{b_2,y_2\},\tfrac{1}{2});(\{e_2,z_2\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2}) * ((\overline{(\{\widehat{y_1}\},\tfrac{1}{2})};(\{\widehat{z_1}\},\tfrac{1}{2}))[]((\{\widehat{y_2}\},\tfrac{1}{2});(\{\widehat{z_2}\},\tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_4 = [([(\{x_1\},\tfrac{1}{2}) * ((\overline{(\{r_1\},\tfrac{1}{2})};(\{b_1,y_1\},\tfrac{1}{2});(\{e_1,z_1\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * ((\{r_2\},\tfrac{1}{2});\overline{(\{b_2,y_2\},\tfrac{1}{2})};(\{e_2,z_2\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2}) * (((\{\widehat{y_1}\},\tfrac{1}{2});(\{\widehat{z_1}\},\tfrac{1}{2}))[](\overline{(\{\widehat{y_2}\},\tfrac{1}{2})};(\{\widehat{z_2}\},\tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_5 = [([(\{x_1\},\tfrac{1}{2}) * ((\{r_1\},\tfrac{1}{2});(\{b_1,y_1\},\tfrac{1}{2});\overline{(\{e_1,z_1\},\tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * (\overline{(\{r_2\},\tfrac{1}{2})};(\{b_2,y_2\},\tfrac{1}{2});(\{e_2,z_2\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2}) * (((\{\widehat{y_1}\},\tfrac{1}{2});\overline{(\{\widehat{z_1}\},\tfrac{1}{2})})[]((\{\widehat{y_2}\},\tfrac{1}{2});(\{\widehat{z_2}\},\tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_6 = [([(\{x_1\},\tfrac{1}{2}) * ((\{r_1\},\tfrac{1}{2});\overline{(\{b_1,y_1\},\tfrac{1}{2})};(\{e_1,z_1\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * ((\{r_2\},\tfrac{1}{2});\overline{(\{b_2,y_2\},\tfrac{1}{2})};(\{e_2,z_2\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2}) * \overline{(((\{\widehat{y_1}\},\tfrac{1}{2});(\{\widehat{z_1}\},\tfrac{1}{2}))[]((\{\widehat{y_2}\},\tfrac{1}{2});(\{\widehat{z_2}\},\tfrac{1}{2})))} * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_7 = [([(\{x_1\}, \tfrac{1}{2}) * (\overline{(\{r_1\}, \tfrac{1}{2})}; (\{b_1, y_1\}, \tfrac{1}{2}); (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{b_2, y_2\}, \tfrac{1}{2}); \overline{(\{e_2, z_2\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_2}\}, \tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_8 = [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); (\{b_1, y_1\}, \tfrac{1}{2}); \overline{(\{e_1, z_1\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); \overline{(\{b_2, y_2\}, \tfrac{1}{2})}; (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_1}\}, \tfrac{1}{2})})[]((\{\widehat{y_2}\}, \tfrac{1}{2}); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_9 = [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); \overline{(\{b_1, y_1\}, \tfrac{1}{2})}; (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{b_2, y_2\}, \tfrac{1}{2}); \overline{(\{e_2, z_2\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_2}\}, \tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx}.$$

Interpretation of the states

s_1: the initial state,

s_2: the system is activated and the memory is not requested,

s_3: the memory is requested by the first processor,

s_4: the memory is requested by the second processor,

s_5: the memory is allocated to the first processor,

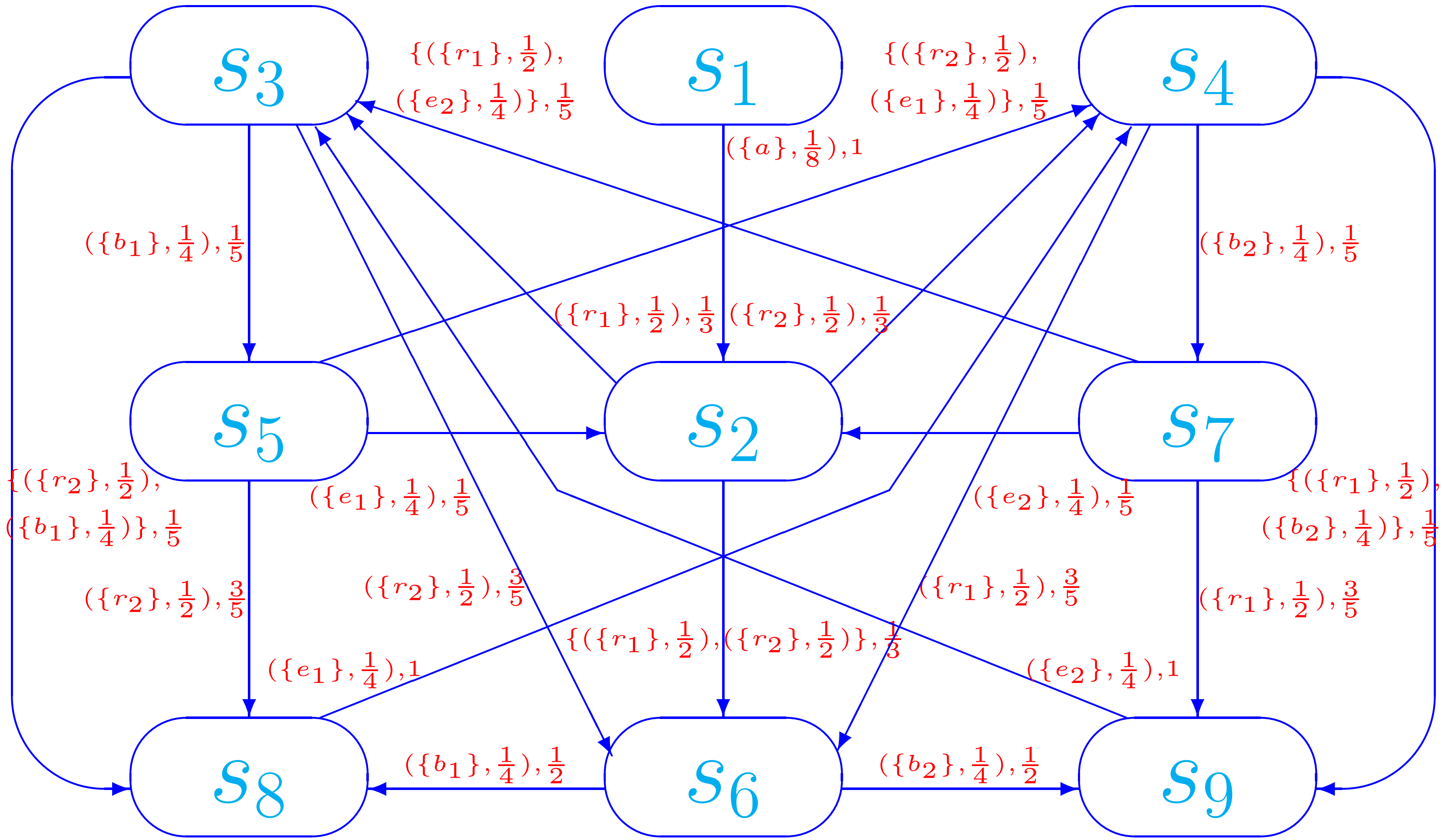
s_6: the memory is requested by two processors,

s_7: the memory is allocated to the second processor,

s_8: the memory is allocated to the first processor and the memory is requested by the second processor,

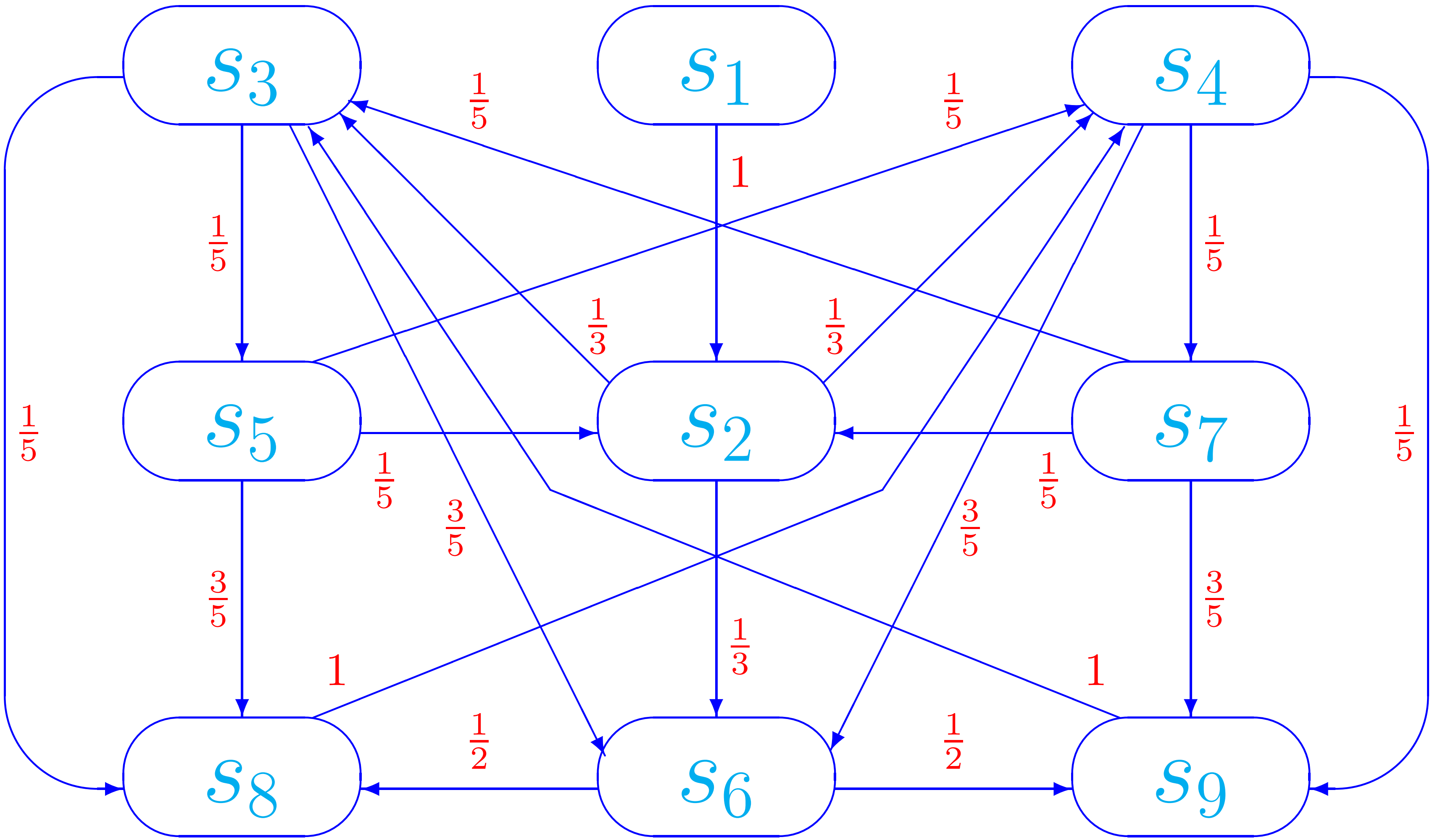
s_9: the memory is allocated to the second processor and the memory is requested by the first processor.

$TS^*(\overline{E})$



The transition system without empty loops of the shared memory system

$DTMC^*(\overline{E})$



The underlying DTMC without empty loops of the shared memory system

The TPM for $DTMC^*(\overline{E})$ is

$$\mathbf{P}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{3}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

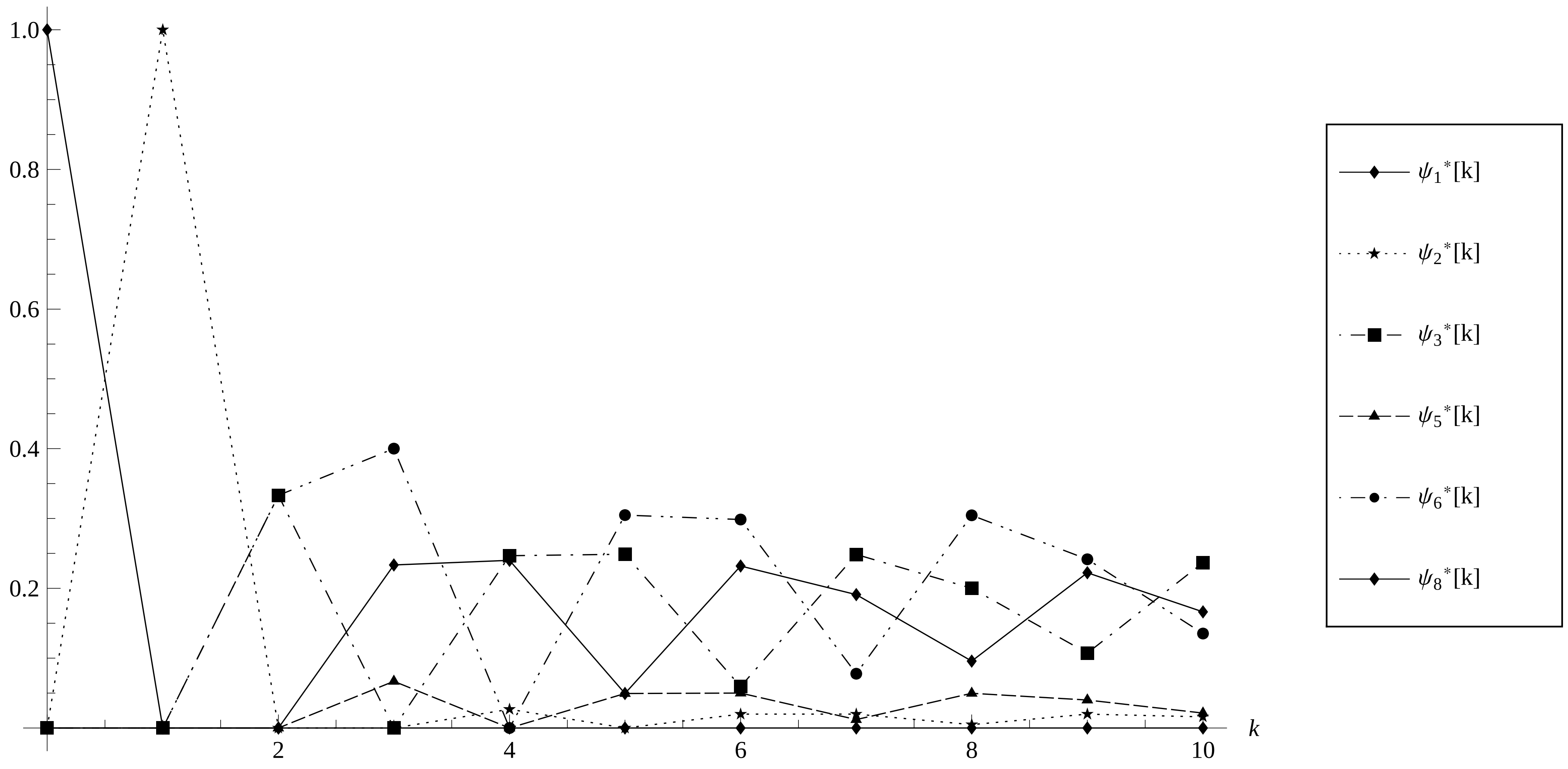
The steady-state PMF for $DTMC^*(\overline{E})$ is

$$\psi^* = \left(0, \frac{3}{209}, \frac{75}{418}, \frac{75}{418}, \frac{15}{418}, \frac{46}{209}, \frac{15}{418}, \frac{35}{209}, \frac{35}{209}\right).$$

Transient and steady-state probabilities of the shared memory system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2^*[k]$	0	1	0	0	0.0267	0	0.0197	0.0199	0.0047	0.0199	0.0160	0.0144
$\psi_3^*[k]$	0	0	0.3333	0	0.2467	0.2489	0.0592	0.2484	0.2000	0.1071	0.2368	0.1794
$\psi_5^*[k]$	0	0	0	0.0667	0	0.0493	0.0498	0.0118	0.0497	0.0400	0.0214	0.0359
$\psi_6^*[k]$	0	0	0.3333	0.4000	0	0.3049	0.2987	0.0776	0.3047	0.2416	0.1351	0.2201
$\psi_8^*[k]$	0	0	0	0.2333	0.2400	0.0493	0.2318	0.1910	0.0956	0.2221	0.1662	0.1675

We depict the probabilities for the states $s_1, s_2, s_3, s_5, s_6, s_8$ only, since the corresponding values coincide for s_3, s_4 as well as for s_5, s_7 as well as for s_8, s_9.



Transient probabilities alteration diagram of the shared memory system

Performance indices

- The average recurrence time in the state s_2, the *average system run-through*, is $\frac{1}{\psi_2^*} = \frac{209}{3} = 69\frac{2}{3}$.
- The common memory is available in the states s_2, s_3, s_4, s_6 only.

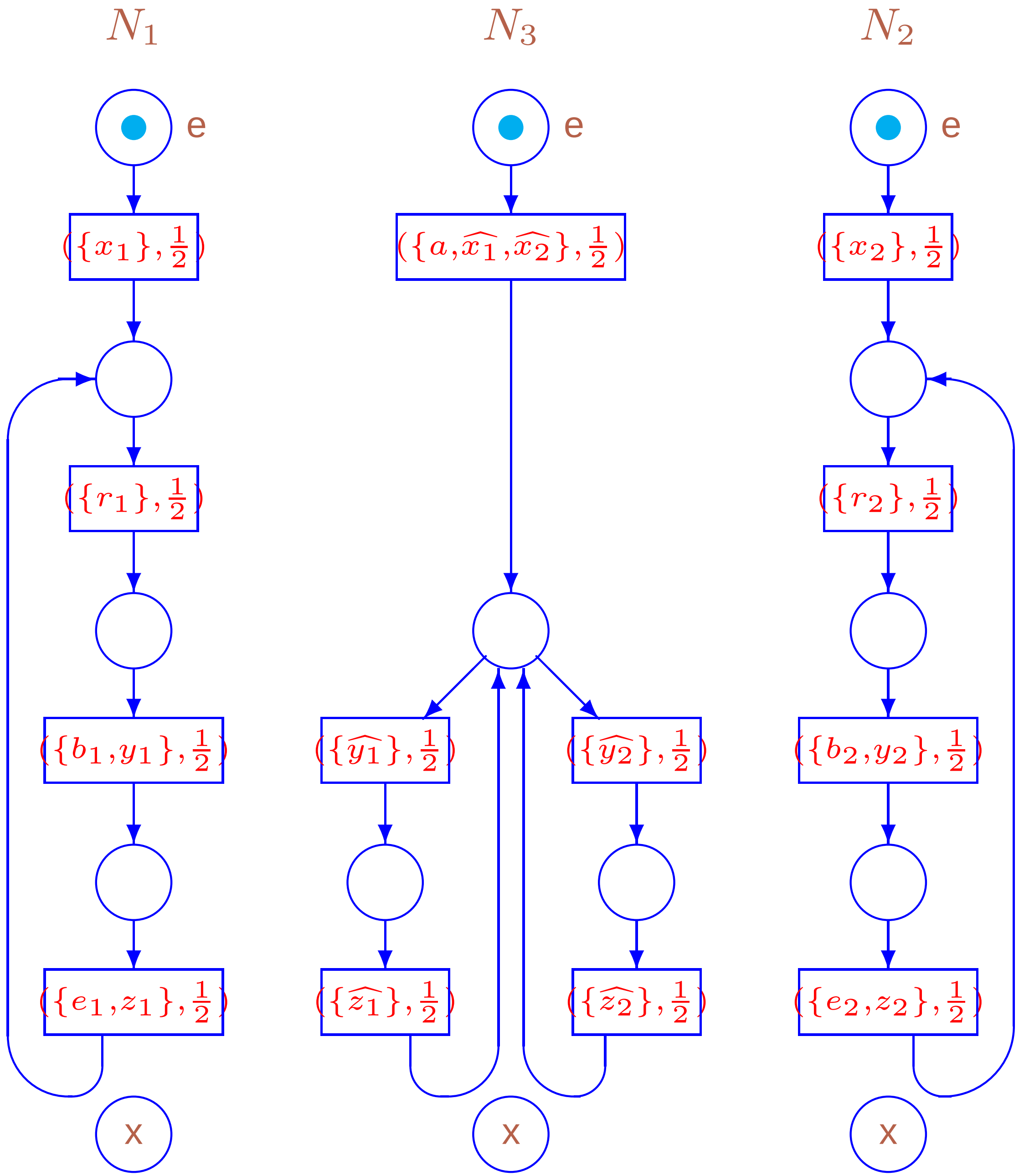
  The steady-state probability that the memory is available is $\psi_2^* + \psi_3^* + \psi_4^* + \psi_6^* = \frac{124}{209}$.

  The steady-state probability that the memory is used, the *shared memory utilization*, is $1 - \frac{124}{209} = \frac{85}{209}$.
- The common memory request of the first processor $(\{r_1\}, \frac{1}{2})$ is only possible from the states s_2, s_4, s_7.

  The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

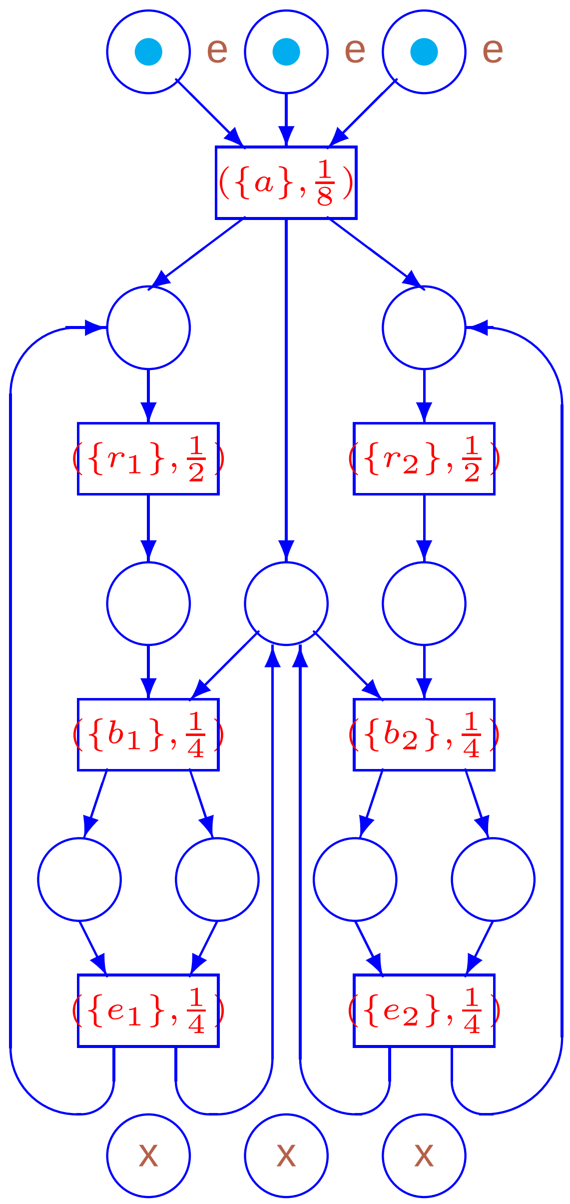
  The *steady-state probability of the shared memory request from the first processor* is

  $$\psi_2^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_4^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_4) + \psi_7^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_7) = \frac{3}{209}\left(\frac{1}{3}+\frac{1}{3}\right) + \frac{75}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{15}{418}\left(\frac{3}{5}+\frac{1}{5}\right) = \frac{38}{209}.$$



The marked dts-boxes of two processors and shared memory

N



The marked dts-box of the shared memory system

The abstract system

The static expression of the first processor is

$F_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{b, y_1\}, \frac{1}{2}); (\{e, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$F_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{b, y_2\}, \frac{1}{2}); (\{e, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is

$F_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, \frac{1}{2}); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \frac{1}{2}); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the abstract shared memory system with two processors is

$F = (F_1 \| F_2 \| F_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS^*(\overline{F})$ is similar to $TS^*(\overline{E})$.

$DTMC^*(\overline{F}) \simeq DTMC^*(\overline{E})$, thus, the TPM and the steady-state PMF for $DTMC^*(\overline{F})$ and $DTMC^*(\overline{E})$ coincide.

Performance indices

The first and second performance indices are the same for the standard and abstract systems.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\}, \frac{1}{2})$ is only possible from the states s_2, s_3, s_4, s_5, s_7.

  The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from a processor* is

$$\psi_2^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_3^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_3) + \psi_4^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_4) + \psi_5^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_5) + \psi_7^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_7) = \frac{3}{209}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) + \frac{75}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{75}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{15}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{15}{418}\left(\frac{3}{5}+\frac{1}{5}\right) = \frac{75}{209}.$$

The quotient of the abstract system

$DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6\}$, where

$\mathcal{K}_1 = \{s_1\}$ (the initial state),

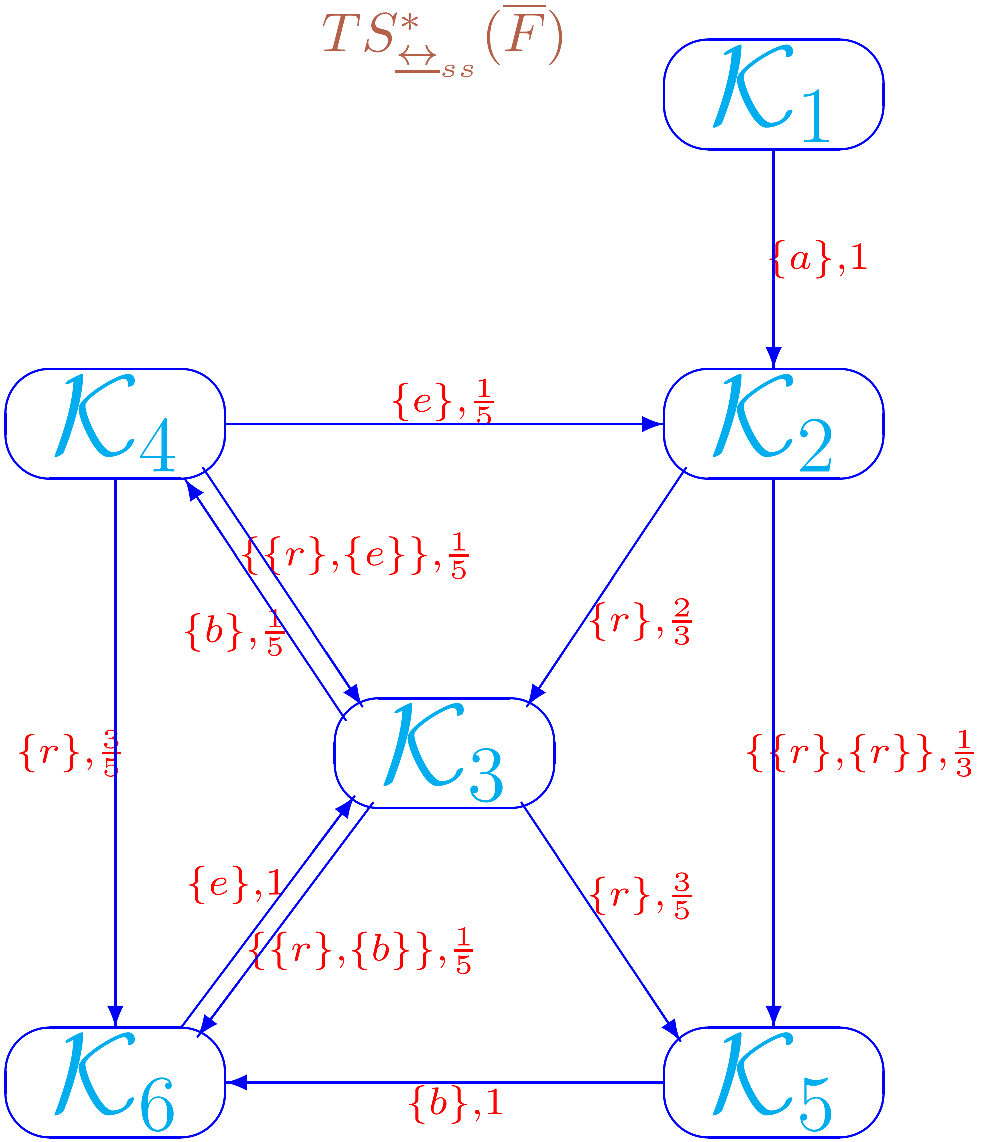
$\mathcal{K}_2 = \{s_2\}$ (the system is activated and the memory is not requested),

$\mathcal{K}_3 = \{s_3, s_4\}$ (the memory is requested by one processor),

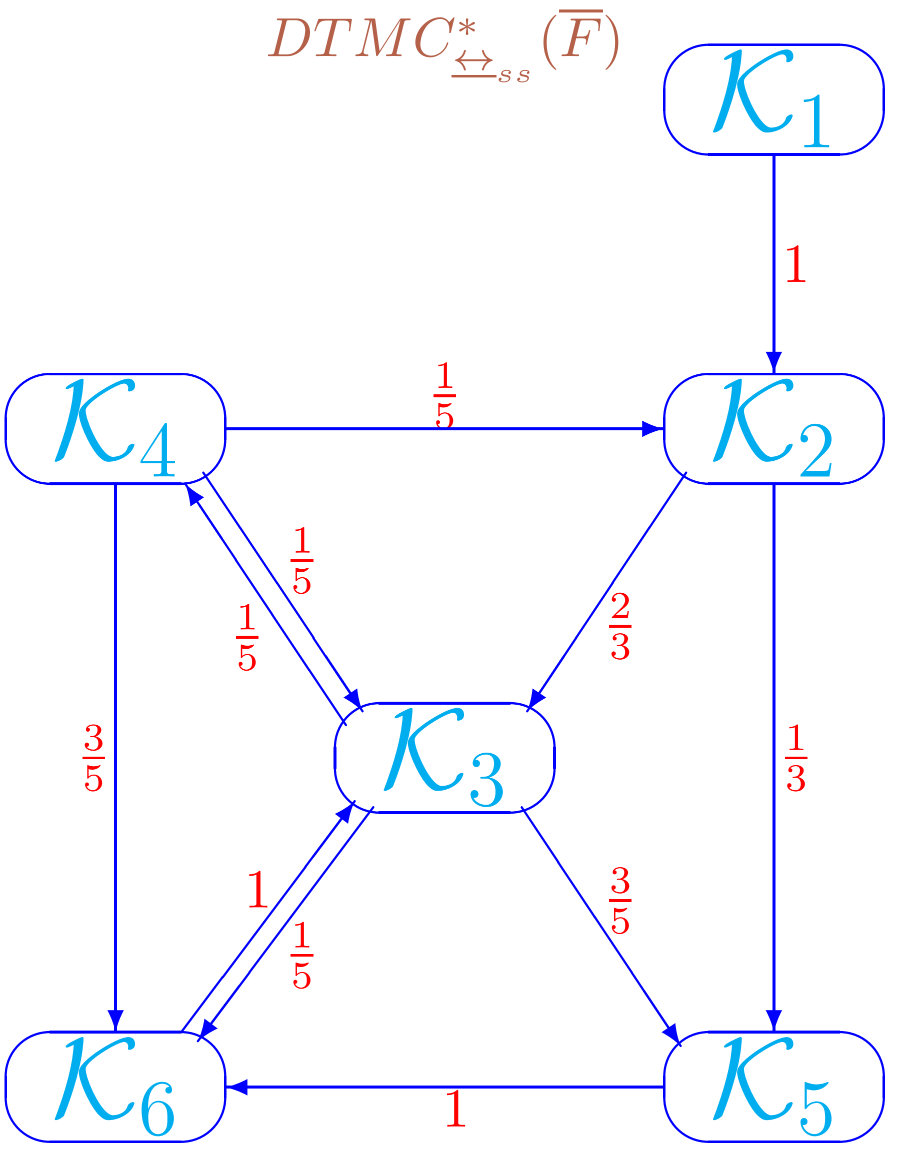
$\mathcal{K}_4 = \{s_5, s_7\}$ (the memory is allocated to a processor),

$\mathcal{K}_5 = \{s_6\}$ (the memory is requested by two processors),

$\mathcal{K}_6 = \{s_8, s_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).



The quotient transition system without empty loops of the abstract shared memory system



The quotient underlying DTMC without empty loops of the abstract shared memory system

The TPM for $DTMC^{*}_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

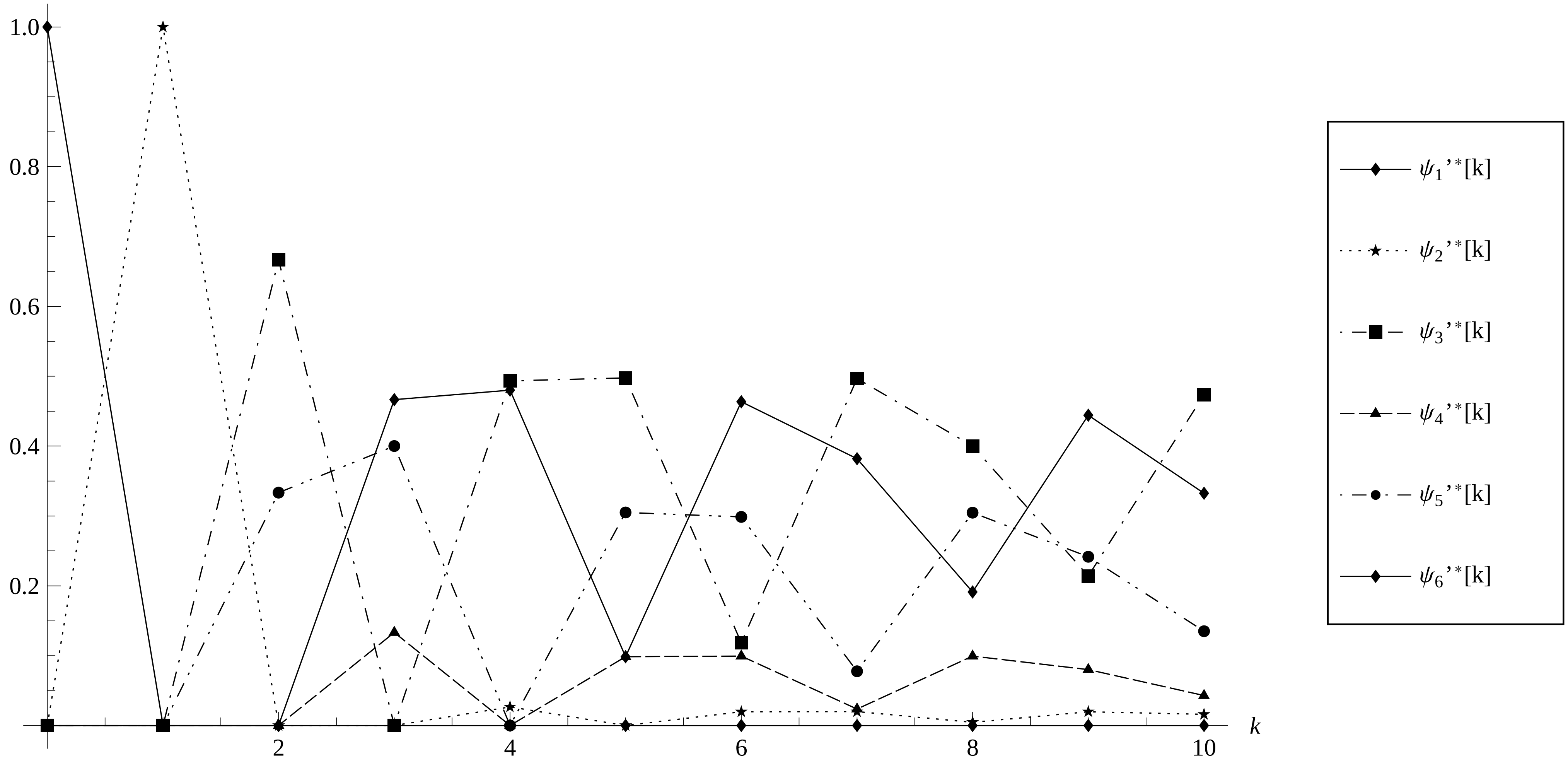
$$\mathbf{P}'^{*} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^{*}_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

$$\psi'^{*} = \left(0, \frac{3}{209}, \frac{75}{209}, \frac{15}{209}, \frac{46}{209}, \frac{70}{209}\right).$$

Transient and steady-state probabilities of the quotient abstract shared memory system

k	0	1	2	3	4	5	6	7	8	9	10	∞
${\psi'_1}^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
${\psi'_2}^*[k]$	0	1	0	0	0.0267	0	0.0197	0.0199	0.0047	0.0199	0.0160	0.0144
${\psi'_3}^*[k]$	0	0	0.6667	0	0.4933	0.4978	0.1184	0.4967	0.4001	0.2142	0.4735	0.3589
${\psi'_4}^*[k]$	0	0	0	0.1333	0	0.0987	0.0996	0.0237	0.0993	0.0800	0.0428	0.0718
${\psi'_5}^*[k]$	0	0	0.3333	0.4000	0	0.3049	0.2987	0.0776	0.3047	0.2416	0.1351	0.2201
${\psi'_6}^*[k]$	0	0	0	0.4667	0.4800	0.0987	0.4636	0.3821	0.1912	0.4443	0.3325	0.3349



Transient probabilities alteration diagram of the quotient abstract shared memory system

Performance indices

- The average recurrence time in the state $\mathcal{K}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{{\psi'_2}^*} = \frac{209}{3} = 69\frac{2}{3}$.
- The common memory is available in the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_5$ only.

  The steady-state probability that the memory is available is ${\psi'_2}^* + {\psi'_3}^* + {\psi'_5}^* = \frac{3}{209} + \frac{75}{209} + \frac{46}{209} = \frac{124}{209}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{124}{209} = \frac{85}{209}$.
- The common memory request of a processor $\{r\}$ is only possible from the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$.

  The request probability in each of the states is a sum of execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is

$$
{\psi'_2}^* \sum_{\{A,\mathcal{K}\mid\{r\}\in A,\ \mathcal{K}_2 \stackrel{A}{\twoheadrightarrow} \mathcal{K}\}} PM_A^*(\mathcal{K}_2, \mathcal{K}) +
{\psi'_3}^* \sum_{\{A,\mathcal{K}\mid\{r\}\in A,\ \mathcal{K}_3 \stackrel{A}{\twoheadrightarrow} \mathcal{K}\}} PM_A^*(\mathcal{K}_3, \mathcal{K}) +
{\psi'_4}^* \sum_{\{A,\mathcal{K}\mid\{r\}\in A,\ \mathcal{K}_4 \stackrel{A}{\twoheadrightarrow} \mathcal{K}\}} PM_A^*(\mathcal{K}_4, \mathcal{K}) =
\frac{3}{209}\left(\frac{2}{3} + \frac{1}{3}\right) + \frac{75}{209}\left(\frac{3}{5} + \frac{1}{5}\right) + \frac{15}{209}\left(\frac{3}{5} + \frac{1}{5}\right) = \frac{75}{209}.
$$

The performance indices are the same for the complete and the quotient abstract shared memory systems.

The coincidence of the first and second performance indices illustrates proposition about steady-state probabilities.

The coincidence of the third performance index theorem about derived step traces from steady states:

one should apply its result to the derived step traces $\{\{r\}\}$, $\{\{r\},\{r\}\}$, $\{\{r\},\{b\}\}$, $\{\{r\},\{e\}\}$ of $\overline{F}$ and itself,

and sum the left and right parts of the three resulting equalities.

The generalized system

The static expression of the first processor is

$$K_1 = [(\{x_1\},\rho) * ((\{r_1\},\rho);(\{b_1,y_1\},\rho);(\{e_1,z_1\},\rho)) * \mathsf{Stop}].$$

The static expression of the second processor is

$$K_2 = [(\{x_2\},\rho) * ((\{r_2\},\rho);(\{b_2,y_2\},\rho);(\{e_2,z_2\},\rho)) * \mathsf{Stop}].$$

The static expression of the shared memory is

$$K_3 = [(\{a,\widehat{x_1},\widehat{x_2}\},\rho) * (((\{\widehat{y_1}\},\rho);(\{\widehat{z_1}\},\rho))[]((\{\widehat{y_2}\},\rho);(\{\widehat{z_2}\},\rho))) * \mathsf{Stop}].$$

The static expression of the generalized shared memory system with two processors is

$$K = (K_1 \| K_2 \| K_3) \ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2.$$

Interpretation of the states

$\tilde{s}_1$: the initial state,

$\tilde{s}_2$: the system is activated and the memory is not requested,

$\tilde{s}_3$: the memory is requested by the first processor,

$\tilde{s}_4$: the memory is requested by the second processor,

$\tilde{s}_5$: the memory is allocated to the first processor,

$\tilde{s}_6$: the memory is requested by two processors,

$\tilde{s}_7$: the memory is allocated to the second processor,

$\tilde{s}_8$: the memory is allocated to the first processor and the memory is requested by the second processor,

$\tilde{s}_9$: the memory is allocated to the second processor and the memory is requested by the first processor.

The TPM for $DTMC^*(\overline{K})$ is

$$\widetilde{\mathbf{P}}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\rho}{2-\rho} & \frac{1-\rho}{2-\rho} & 0 & \frac{\rho}{2-\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{1-\rho^2}{1+\rho-\rho^2} & 0 & \frac{\rho^2}{1+\rho-\rho^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & 0 & \frac{\rho^2}{1+\rho-\rho^2} \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & 0 & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^*(\overline{K})$ is

$$\tilde{\psi}^* = \frac{1}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}(0, 2\rho^2(2-\rho)(1-\rho)^2, (2-p)(1-p+p^2)^2,$$
$$(2-p)(1-p+p^2)^2, \rho(2-\rho-4\rho^2+4\rho^3-\rho^4), 2(2+\rho-5\rho^2+\rho^3+\rho^4),$$
$$\rho(2-\rho-4\rho^2+4\rho^3-\rho^4), 2+3\rho-6\rho^2+\rho^3+\rho^4, 2+3\rho-6\rho^2+\rho^3+\rho^4).$$

Performance indices

- The average recurrence time in the state $\tilde{s}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\tilde{\psi}_2^*} = \frac{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}{\rho^2(2-\rho)(1-\rho)^2}$.
- The common memory is available only in the states $\tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_6$.

  The steady-state probability that the memory is available is $\tilde{\psi}_2^* + \tilde{\psi}_3^* + \tilde{\psi}_4^* + \tilde{\psi}_6^* = \frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} + \frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)} + \frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)} + \frac{2+\rho-5\rho^2+\rho^3+\rho^4}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} = \frac{4+4\rho-7\rho^2-7\rho^3+9\rho^4-2\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{4+4\rho-7\rho^2-7\rho^3+9\rho^4-2\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} = \frac{2+5\rho-7\rho^2-3\rho^3+5\rho^4-\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

- The common memory request of the first processor $(\{r_1\},\rho)$ is only possible from the states $\tilde{s}_2, \tilde{s}_4, \tilde{s}_7$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r_1\},\rho)$.

  The *steady-state probability of the shared memory request from the first processor* is

$$\tilde{\psi}_2^* \sum_{\{\Gamma|(\{r_1\},\rho)\in\Gamma\}} PT^*(\Gamma,\tilde{s}_2) + \tilde{\psi}_4^* \sum_{\{\Gamma|(\{r_1\},\rho)\in\Gamma\}} PT^*(\Gamma,\tilde{s}_4) + \tilde{\psi}_7^* \sum_{\{\Gamma|(\{r_1\},\rho)\in\Gamma\}} PT^*(\Gamma,\tilde{s}_7) =$$

$$\frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{1-\rho}{2-\rho}+\frac{\rho}{2-\rho}\right) + \frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right) + \frac{\rho(2-\rho-4\rho^2+4\rho^3-\rho^4)}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right) = \frac{2+3\rho-4\rho^2-2\rho^3+2\rho^4}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}.$$

The abstract generalized system and its reduction

The static expression of the first processor is

$L_1 = [(\{x_1\}, \rho) * ((\{r\}, \rho); (\{b, y_1\}, \rho); (\{e, z_1\}, \rho)) * \mathsf{Stop}]$.

The static expression of the second processor is

$L_2 = [(\{x_2\}, \rho) * ((\{r\}, \rho); (\{b, y_2\}, \rho); (\{e, z_2\}, \rho)) * \mathsf{Stop}]$.

The static expression of the shared memory is

$L_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \rho) * (((\{\widehat{y_1}\}, \rho); (\{\widehat{z_1}\}, \rho))[]((\{\widehat{y_2}\}, \rho); (\{\widehat{z_2}\}, \rho))) * \mathsf{Stop}]$.

The static expression of the abstract shared memory generalized system with two processors is

$L = (L_1 \| L_2 \| L_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$DR(\overline{L})$ resembles $DR(\overline{K})$, and $TS^*(\overline{L})$ is similar to $TS^*(\overline{K})$.

$DTMC^*(\overline{L}) \simeq DTMC^*(\overline{K})$, thus, the TPM and the steady-state PMF for $DTMC^*(\overline{L})$ and $DTMC^*(\overline{K})$ coincide.

Performance indices

The first and second performance indices are the same for the generalized system and its abstraction.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\},\rho)$ is only possible from the states $\tilde{s}_2,\tilde{s}_3,\tilde{s}_4,\tilde{s}_5,\tilde{s}_7$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r\},\rho)$.

  The *steady-state probability of the shared memory request from a processor* is

$$\tilde{\psi}_2^*\sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}}PT^*(\Gamma,\tilde{s}_2)+\tilde{\psi}_3^*\sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}}PT^*(\Gamma,\tilde{s}_3)+$$
$$\tilde{\psi}_4^*\sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}}PT^*(\Gamma,\tilde{s}_4)+\tilde{\psi}_5^*\sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}}PT^*(\Gamma,\tilde{s}_5)+$$
$$\tilde{\psi}_7^*\sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}}PT^*(\Gamma,\tilde{s}_7)=\frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{1-\rho}{2-\rho}+\frac{1-\rho}{2-\rho}+\frac{\rho}{2-\rho}\right)+$$
$$\frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right)+$$
$$\frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right)+$$
$$\frac{\rho(2-\rho-4\rho^2+4\rho^3-\rho^4)}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right)+$$
$$\frac{\rho(2-\rho-4\rho^2+4\rho^3-\rho^4)}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right)=\frac{(2-\rho)(1+\rho-\rho^2)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}.$$

The quotient of the abstract system

$DR(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_5, \widetilde{\mathcal{K}}_6\}$, where

$\widetilde{\mathcal{K}}_1 = \{\tilde{s}_1\}$ (the initial state),

$\widetilde{\mathcal{K}}_2 = \{\tilde{s}_2\}$ (the system is activated and the memory is not requested),

$\widetilde{\mathcal{K}}_3 = \{\tilde{s}_3, \tilde{s}_4\}$ (the memory is requested by one processor),

$\widetilde{\mathcal{K}}_4 = \{\tilde{s}_5, \tilde{s}_7\}$ (the memory is allocated to a processor),

$\widetilde{\mathcal{K}}_5 = \{\tilde{s}_6\}$ (the memory is requested by two processors),

$\widetilde{\mathcal{K}}_6 = \{\tilde{s}_8, \tilde{s}_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).

The TPM for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{L})$ is

$$\widetilde{\mathbf{P}}'^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2(1-\rho)}{2-\rho} & 0 & \frac{\rho}{2-\rho} & 0 \\ 0 & 0 & 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{1-\rho^2}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{L})$ is

$$\tilde{\psi}'^* = \frac{1}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}(0, \rho^2(2-\rho)(1-\rho)^2, (2-\rho)(1+\rho-\rho^2)^2,$$
$$\rho(2-\rho-4\rho^2+4\rho^3-\rho^4), 2+\rho-5\rho^2+\rho^3+\rho^4, 2+3\rho-6\rho^2+\rho^3+\rho^4).$$

Performance indices

- The average recurrence time in the state $\widetilde{\mathcal{K}}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\tilde{\psi}_2'^*} = \frac{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}{\rho^2(2-\rho)(1-\rho)^2}$.

- The common memory is available only in the states $\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5$.

  The steady-state probability that the memory is available is $\tilde{\psi}_2'^* + \tilde{\psi}_3'^* + \tilde{\psi}_5'^* = \frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} + \frac{(2-\rho)(1+\rho-\rho^2)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} + \frac{2+\rho-5\rho^2+\rho^3+\rho^4}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} = \frac{4+4\rho-7\rho^2-7\rho^3+9\rho^4-2\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{4+4\rho-7\rho^2-7\rho^3+9\rho^4-2\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} = \frac{2+5\rho-7\rho^2-3\rho^3+5\rho^4-\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

- The common memory request of a processor $\{r\}$ is only possible from the states $\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is

$$\tilde{\psi}_2'^* \sum_{\{A,\widetilde{\mathcal{K}} \mid \{r\}\in A,\ \widetilde{\mathcal{K}}_2 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}) + \tilde{\psi}_3'^* \sum_{\{A,\widetilde{\mathcal{K}} \mid \{r\}\in A,\ \widetilde{\mathcal{K}}_3 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}) + \tilde{\psi}_4'^* \sum_{\{A,\widetilde{\mathcal{K}} \mid \{r\}\in A,\ \widetilde{\mathcal{K}}_4 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}) =$$

$$\frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{2(1-\rho)}{2-\rho}+\frac{\rho}{2-\rho}\right) + \frac{(2-\rho)(1+\rho-\rho^2)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right) + \frac{\rho(2-\rho-4\rho^2+4\rho^3-\rho^4)}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right) = \frac{(2-\rho)(1+\rho-\rho^2)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}.$$

The performance indices are the same for the complete and the quotient abstract generalized shared memory systems.

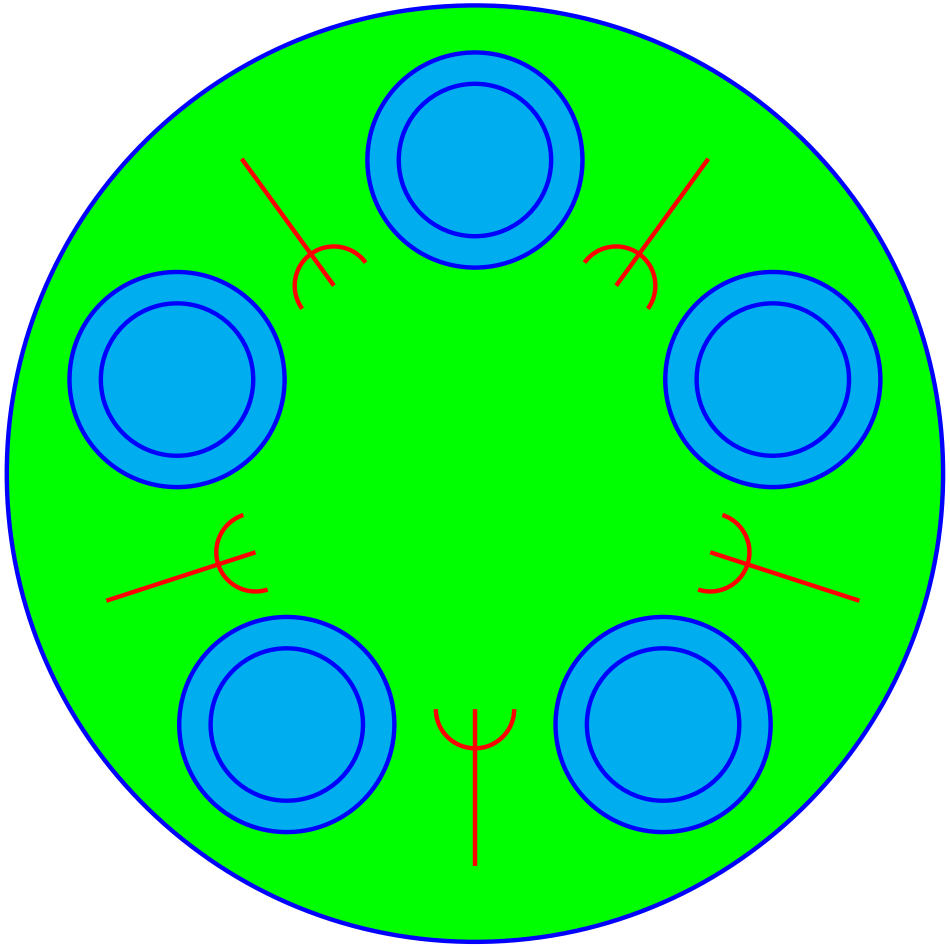
The coincidence of the first and second performance indices illustrates proposition about steady-state probabilities.

The coincidence of the third performance index theorem about derived step traces from steady states: one should apply its result to the derived step traces $\{\{r\}\},\ \{\{r\},\{r\}\},\ \{\{r\},\{b\}\},\ \{\{r\},\{e\}\}$ of $\overline{L}$ and itself, and sum the left and right parts of the three resulting equalities.

Dining philosophers system

The standard system

A model of five dining philosophers [P81]



The diagram of the dining philosophers system

Arbitrary number of philosophers

The most interesting: the maximal sets of philosophers which can dine together.

The system with 1 philosopher: the only maximal set is $\emptyset$.

The system with 2 philosophers: the maximal sets are $\{1\}$, $\{2\}$.

The system with 3 philosophers: the maximal sets are $\{1\}$, $\{2\}$, $\{3\}$.

The system with 4 philosophers: the maximal sets are $\{1, 3\}$, $\{2, 4\}$.

The system with 5 philosophers: the maximal sets are $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 5\}$.

The system with 6 philosophers: the maximal sets are $\{1, 4\}$, $\{2, 5\}$, $\{3, 6\}$, $\{1, 3, 5\}$, $\{2, 4, 6\}$.

The system with 7 philosophers: the maximal sets are
$\{1, 3, 5\}$, $\{1, 3, 6\}$, $\{1, 4, 6\}$, $\{2, 4, 6\}$, $\{2, 4, 7\}$, $\{2, 5, 7\}$, $\{3, 5, 7\}$.

A nontrivial behaviour: at least 5 philosophers occupy the table.

The neighbors cannot dine together: the maximal number of the dining persons for the system with n philosophers will be $\lfloor \frac{n}{2} \rfloor$.

If the philosopher i belongs to some maximal set then the philosopher $i (\text{mod } n) + 1$ belongs to the next one.

- n is an even number: 2 maximal sets of $\frac{n}{2}$ persons,

  i.e. the philosophers numbered with all odd natural numbers $\leq n$

  and those numbered with all even natural numbers $\leq n$.

- n is an odd number: n maximal sets of $\frac{n-1}{2}$ persons,

  since from a maximal set one can “shift” clockwise $n - 1$ times by one element modulo n until the next maximal set will coincide with the initial one.

After activation of the system (the philosophers come in the dining room), five forks appear on the table.

If the left and right forks available for a philosopher, he takes them simultaneously and begins eating.

At the end of eating, the philosopher places both his forks simultaneously back on the table.

a corresponds to the system activation.

b_i and e_i correspond to the beginning and the end of eating of philosopher i $(1 \leq i \leq 5)$.

The other actions are used for communication purpose only.

The expression of each philosopher includes two alternative subexpressions:
the second one specifies a resource (fork) sharing with the right neighbor.

The static expression of the philosopher i $(1 \leq i \leq 4)$ is

$E_i = [(\{x_i\}, \frac{1}{2}) * ((((\{b_i, \widehat{y_i}\}, \frac{1}{2}); (\{e_i, \widehat{z_i}\}, \frac{1}{2}))[]((\{y_{i+1}\}, \frac{1}{2}); (\{z_{i+1}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is

$E_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * ((((\{b_5, \widehat{y_5}\}, \frac{1}{2}); (\{e_5, \widehat{z_5}\}, \frac{1}{2}))[]((\{y_1\}, \frac{1}{2}); (\{z_1\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the dining philosophers system is

$E = (E_1 \| E_2 \| E_3 \| E_4 \| E_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

Effect of synchronization

Synchronization of $(\{b_i, y_i\}, \frac{1}{2})$ and $(\{\widehat{y_i}\}, \frac{1}{2})$ produces $(\{b_i\}, \frac{1}{4})$ $(1 \leq i \leq 5)$.

Synchronization of $(\{e_i, z_i\}, \frac{1}{2})$ and $(\{\widehat{z_i}\}, \frac{1}{2})$ produces $(\{e_i\}, \frac{1}{4})$ $(1 \leq i \leq 5)$.

Synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{2})$ and $(\{x_1\}, \frac{1}{2})$ produces $(\{a, \widehat{x_2}, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{4})$.

Synchronization of $(\{a, \widehat{x_2}, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{4})$ and $(\{x_2\}, \frac{1}{2})$ produces $(\{a, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{8})$.

Synchronization of $(\{a, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{8})$ and $(\{x_3\}, \frac{1}{2})$ produces $(\{a, \widehat{x_4}\}, \frac{1}{16})$.

Synchronization of $(\{a, \widehat{x_4}\}, \frac{1}{16})$ and $(\{x_4\}, \frac{1}{2})$ produces $(\{a\}, \frac{1}{32})$.

$DR(\overline{E})$ consists of

$$s_1 = [([\overline{(\{x_1\}, \tfrac{1}{2})} * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[\overline{(\{x_2\}, \tfrac{1}{2})} * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[\overline{(\{x_3\}, \tfrac{1}{2})} * (((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[\overline{(\{x_4\}, \tfrac{1}{2})} * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[\overline{(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2})} * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_2 = [([(\{x_1\}, \tfrac{1}{2}) * \overline{(((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * \overline{(((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * \overline{(((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * \overline{(((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * \overline{(((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2})))} * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_3 = [([(\{x_1\}, \tfrac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{e_1, \widehat{z_1}\}, \tfrac{1}{2})})[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2}))) * \text{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[](\overline{(\{y_3\}, \tfrac{1}{2})}; (\{z_3\}, \tfrac{1}{2}))) * \text{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (\overline{((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2}))}) * \text{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * ((\overline{(\{b_4, \widehat{y_4}\}, \tfrac{1}{2})}; (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2}))) * \text{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); \overline{(\{z_1\}, \tfrac{1}{2})})) * \text{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_4 = [([(\{x_1\}, \tfrac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{e_1, \widehat{z_1}\}, \tfrac{1}{2})})[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2}))) * \text{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (\overline{((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2}))}) * \text{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); \overline{(\{z_4\}, \tfrac{1}{2})})) * \text{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); \overline{(\{e_4, \widehat{z_4}\}, \tfrac{1}{2})})[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2}))) * \text{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); \overline{(\{z_1\}, \tfrac{1}{2})})) * \text{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_5 = [([(\{x_1\}, \tfrac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{e_1, \widehat{z_1}\}, \tfrac{1}{2})})[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); \overline{(\{z_3\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); \overline{(\{e_3, \widehat{z_3}\}, \tfrac{1}{2})})[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * \overline{(((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); \overline{(\{z_1\}, \tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_6 = [([(\{x_1\}, \tfrac{1}{2}) * \overline{(((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * ((\overline{(\{b_2, \widehat{y_2}\}, \tfrac{1}{2})}; (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); \overline{(\{z_4\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); \overline{(\{e_4, \widehat{z_4}\}, \tfrac{1}{2})})[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[](\overline{(\{y_1\}, \tfrac{1}{2})}; (\{z_1\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_7 = [([(\{x_1\}, \tfrac{1}{2}) * (((\overline{(\{b_1, \widehat{y_1}\}, \tfrac{1}{2})}; (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); \overline{(\{z_3\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); \overline{(\{e_3, \widehat{z_3}\}, \tfrac{1}{2})})[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[](\overline{(\{y_5\}, \tfrac{1}{2})}; (\{z_5\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * \overline{(((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2})))} * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_8 = [([(\{x_1\}, \tfrac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\{y_2\}, \tfrac{1}{2}); \overline{(\{z_2\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{e_2, \widehat{z_2}\}, \tfrac{1}{2})})[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); \overline{(\{z_4\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); \overline{(\{e_4, \widehat{z_4}\}, \tfrac{1}{2})})[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * \overline{(((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2})))} * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_9 = [([(\{x_1\},\tfrac{1}{2}) * \overline{(((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});(\{z_2\},\tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * (((\{b_2,\widehat{y_2}\},\tfrac{1}{2});(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[]((\{y_3\},\tfrac{1}{2});\overline{(\{z_3\},\tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_3\},\tfrac{1}{2}) * (((\{b_3,\widehat{y_3}\},\tfrac{1}{2});\overline{(\{e_3,\widehat{z_3}\},\tfrac{1}{2})})[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_4\},\tfrac{1}{2}) * (((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});\overline{(\{z_5\},\tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2}) * (((\{b_5,\widehat{y_5}\},\tfrac{1}{2});\overline{(\{e_5,\widehat{z_5}\},\tfrac{1}{2})})[]((\{y_1\},\tfrac{1}{2});(\{z_1\},\tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_\approx,$$

$$s_{10} = [([(\{x_1\},\tfrac{1}{2}) * (((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});\overline{(\{z_2\},\tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * (((\{b_2,\widehat{y_2}\},\tfrac{1}{2});\overline{(\{e_2,\widehat{z_2}\},\tfrac{1}{2})})[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_3\},\tfrac{1}{2}) * (((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[](\overline{(\{y_4\},\tfrac{1}{2})};(\{z_4\},\tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_4\},\tfrac{1}{2}) * \overline{(((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2}) * ((\overline{(\{b_5,\widehat{y_5}\},\tfrac{1}{2})};(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2});(\{z_1\},\tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_\approx,$$

$$s_{11} = [([(\{x_1\}, \tfrac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\overline{(\{y_2\}, \tfrac{1}{2})}; (\{z_2\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * \overline{(((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * ((\overline{(\{b_3, \widehat{y_3}\}, \tfrac{1}{2})}; (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); \overline{(\{z_5\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); \overline{(\{e_5, \widehat{z_5}\}, \tfrac{1}{2})})[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_{12} = [([(\{x_1\}, \tfrac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\{y_2\}, \tfrac{1}{2}); \overline{(\{z_2\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{e_2, \widehat{z_2}\}, \tfrac{1}{2})})[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * \overline{(((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); \overline{(\{z_5\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); \overline{(\{e_5, \widehat{z_5}\}, \tfrac{1}{2})})[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx}.$$

Interpretation of the states

s_1: the initial state,

s_2: the system is activated and no philosophers dine,

s_3: philosopher 1 dines,

s_4: philosophers 1 and 4 dine,

s_5: philosophers 1 and 3 dine,

s_6: philosopher 4 dines,

s_7: philosopher 3 dines,

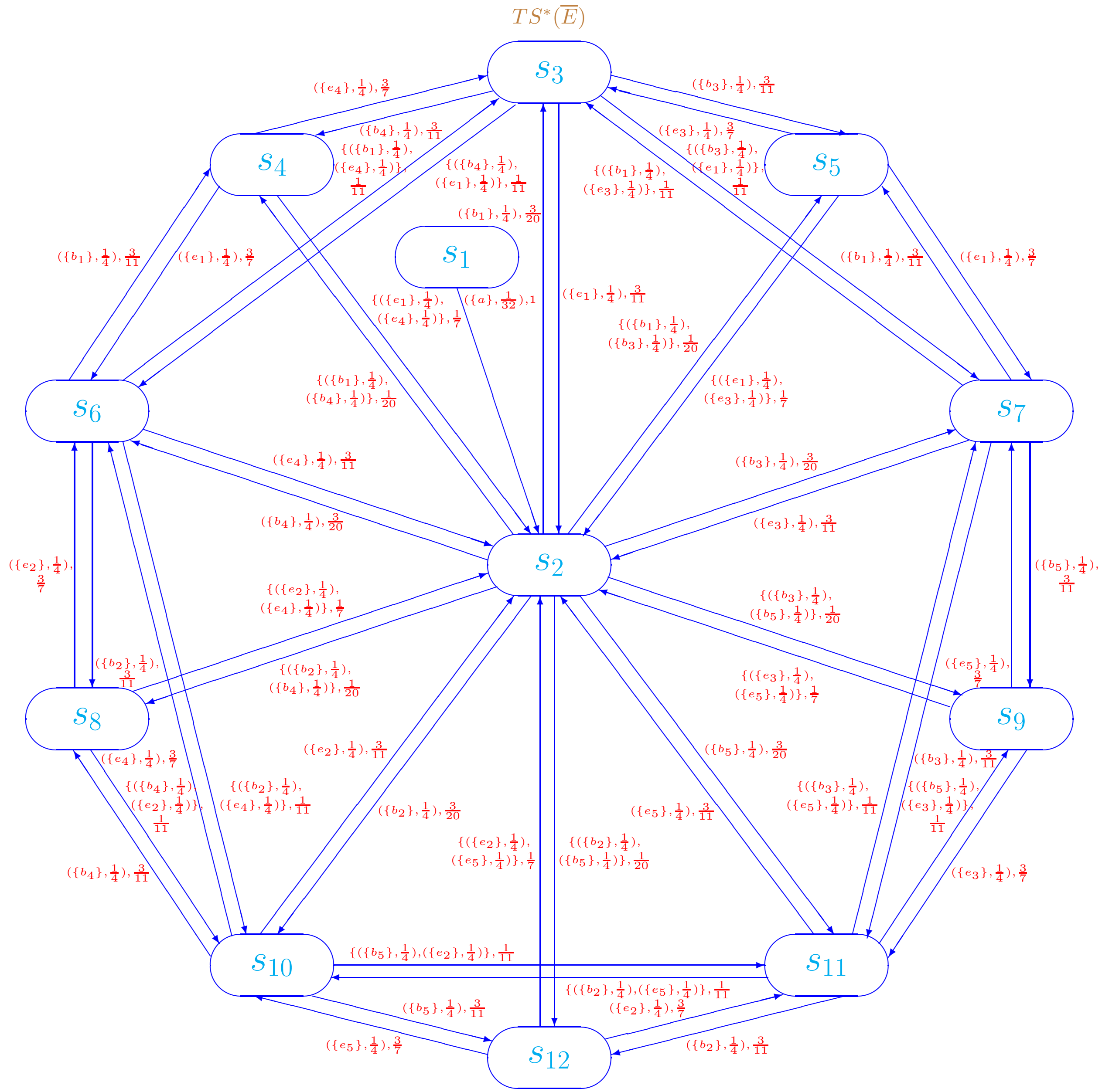
s_8: philosophers 2 and 4 dine,

s_9: philosophers 3 and 5 dine,

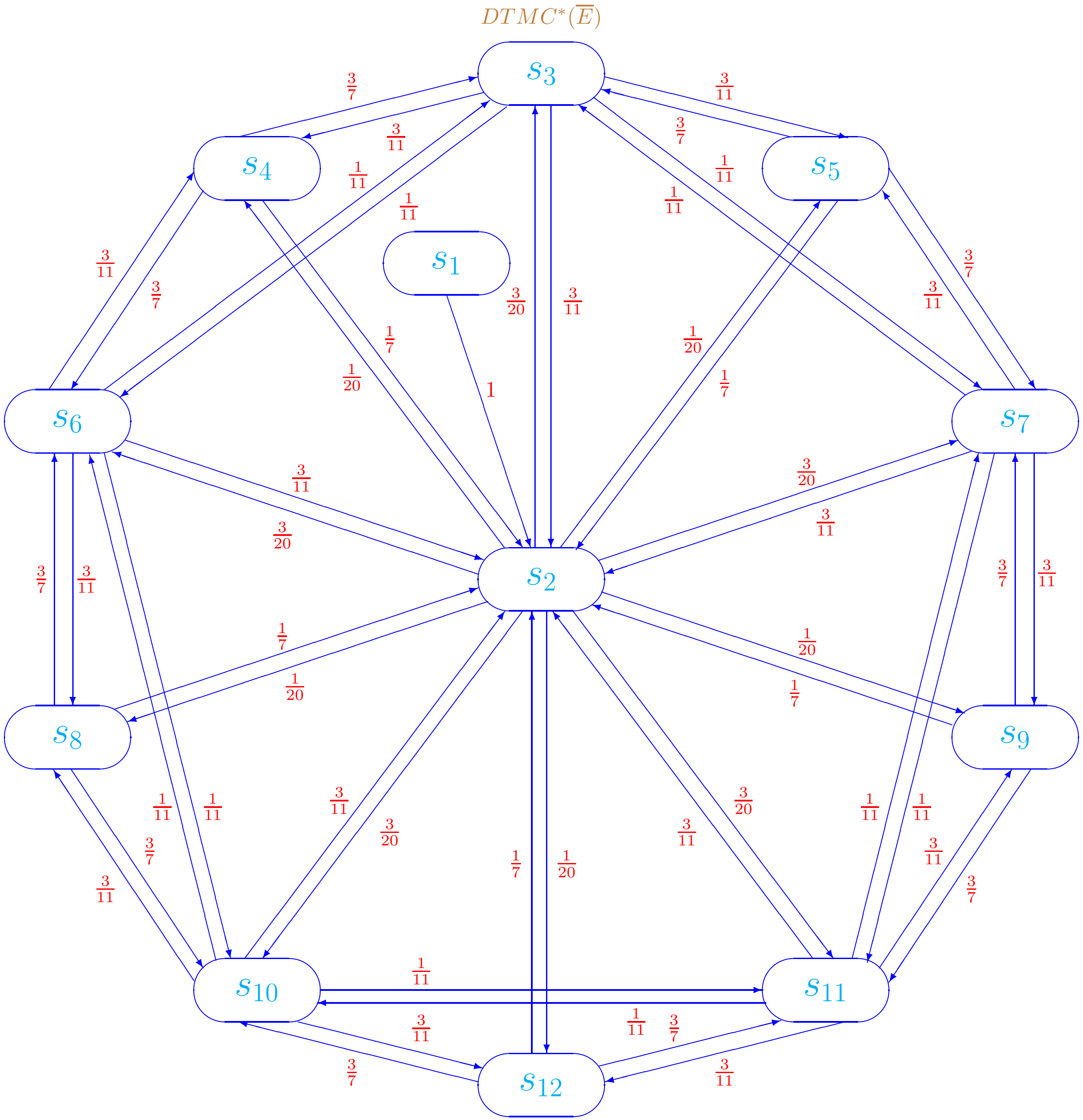
s_{10}: philosopher 2 dines,

s_{11}: philosopher 5 dine,

s_{12}: philosophers 2 and 5 dine.



The transition system without empty loops of the dining philosophers system



The underlying DTMC without empty loops of the dining philosophers system

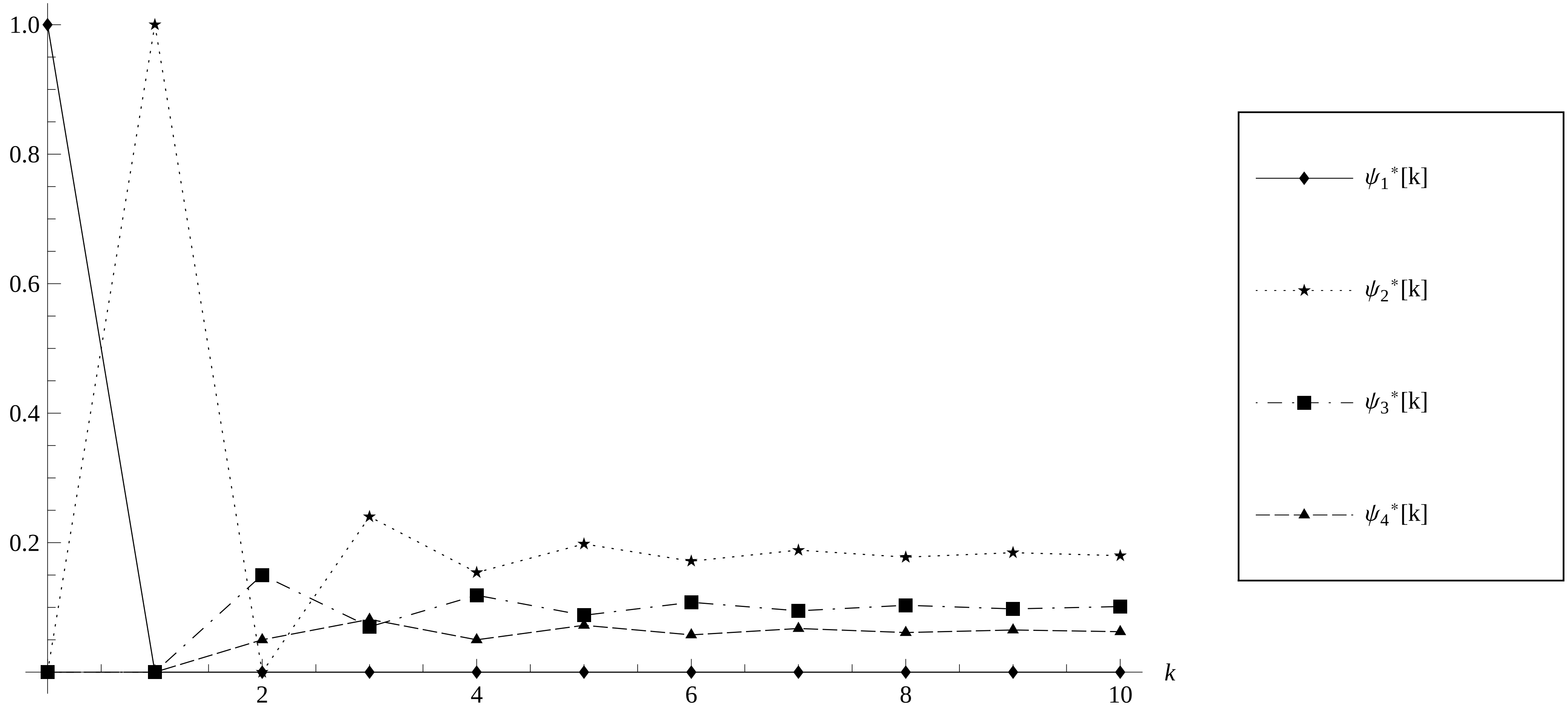
The TPM for $DTMC^*(\overline{E})$ is

$$\mathbf{P}^* = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\
0 & \frac{3}{11} & 0 & \frac{3}{11} & \frac{3}{11} & \frac{1}{11} & \frac{1}{11} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{11} & \frac{1}{11} & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 & 0 \\
0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 \\
0 & \frac{1}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 \\
0 & \frac{1}{7} & 0 & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 \\
0 & \frac{3}{11} & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & \frac{1}{11} & \frac{3}{11} \\
0 & \frac{3}{11} & 0 & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} \\
0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{7} & \frac{3}{7} & 0
\end{pmatrix}.$$

Transient and steady-state probabilities of the dining philosophers system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2^*[k]$	0	1	0	0.2403	0.1541	0.1981	0.1716	0.1884	0.1776	0.1846	0.1800	0.1818
$\psi_3^*[k]$	0	0	0.1500	0.0701	0.1189	0.0878	0.1079	0.0949	0.1033	0.0979	0.1014	0.1000
$\psi_4^*[k]$	0	0	0.0500	0.0818	0.0503	0.0726	0.0578	0.0674	0.0612	0.0652	0.0626	0.0636

We depict the probabilities for the states $s_1, \ldots, s_4$ only, since the corresponding values coincide for the states $s_3, s_6, s_7, s_{10}, s_{11}$ as well as for $s_4, s_5, s_8, s_9, s_{12}$.



Transient probabilities alteration diagram of the dining philosophers system

The steady-state PMF for $DTMC^*(\overline{E})$ is

$$\psi^* = \left(0, \frac{2}{11}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}\right).$$

Performance indices

- The average recurrence time in the state s_2, where all the forks are available, the *average system run-through*, is $\frac{1}{\psi_2^*} = \frac{11}{2} = 5\frac{1}{2}$.

- Nobody eats in the state s_2. The *fraction of time when no philosophers dine* is $\psi_2^* = \frac{2}{11}$.

  Only one philosopher eats in the states $s_3, s_6, s_7, s_{10}, s_{11}$. The *fraction of time when only one philosopher dines* is $\psi_3^* + \psi_6^* + \psi_7^* + \psi_{10}^* + \psi_{11}^* = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{1}{2}$.

  Two philosophers eat together in the states $s_4, s_5, s_8, s_9, s_{12}$. The *fraction of time when two philosophers dine* is $\psi_4^* + \psi_5^* + \psi_8^* + \psi_9^* + \psi_{12}^* = \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} = \frac{7}{22}$.

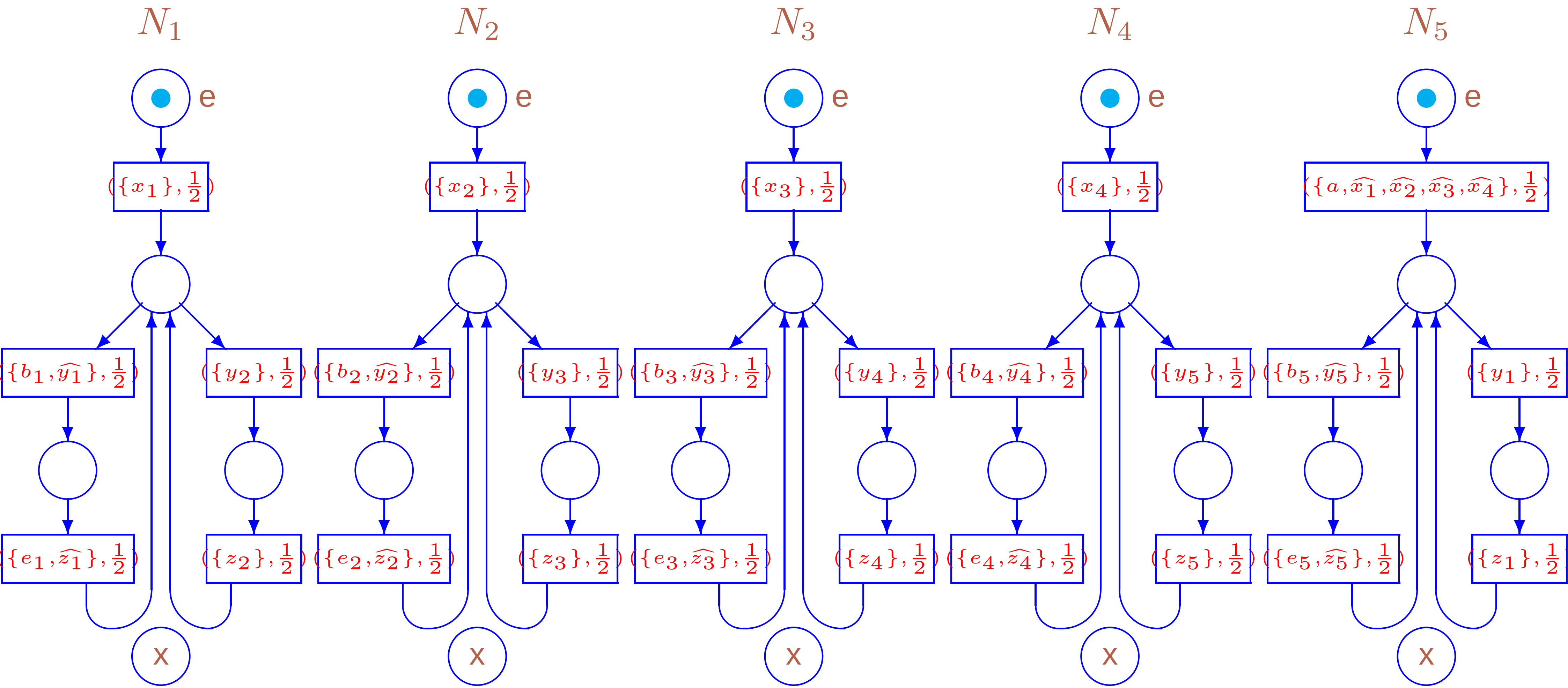
  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.

- The beginning of eating of first philosopher $(\{b_1\}, \frac{1}{4})$ is only possible from the states s_2, s_6, s_7.

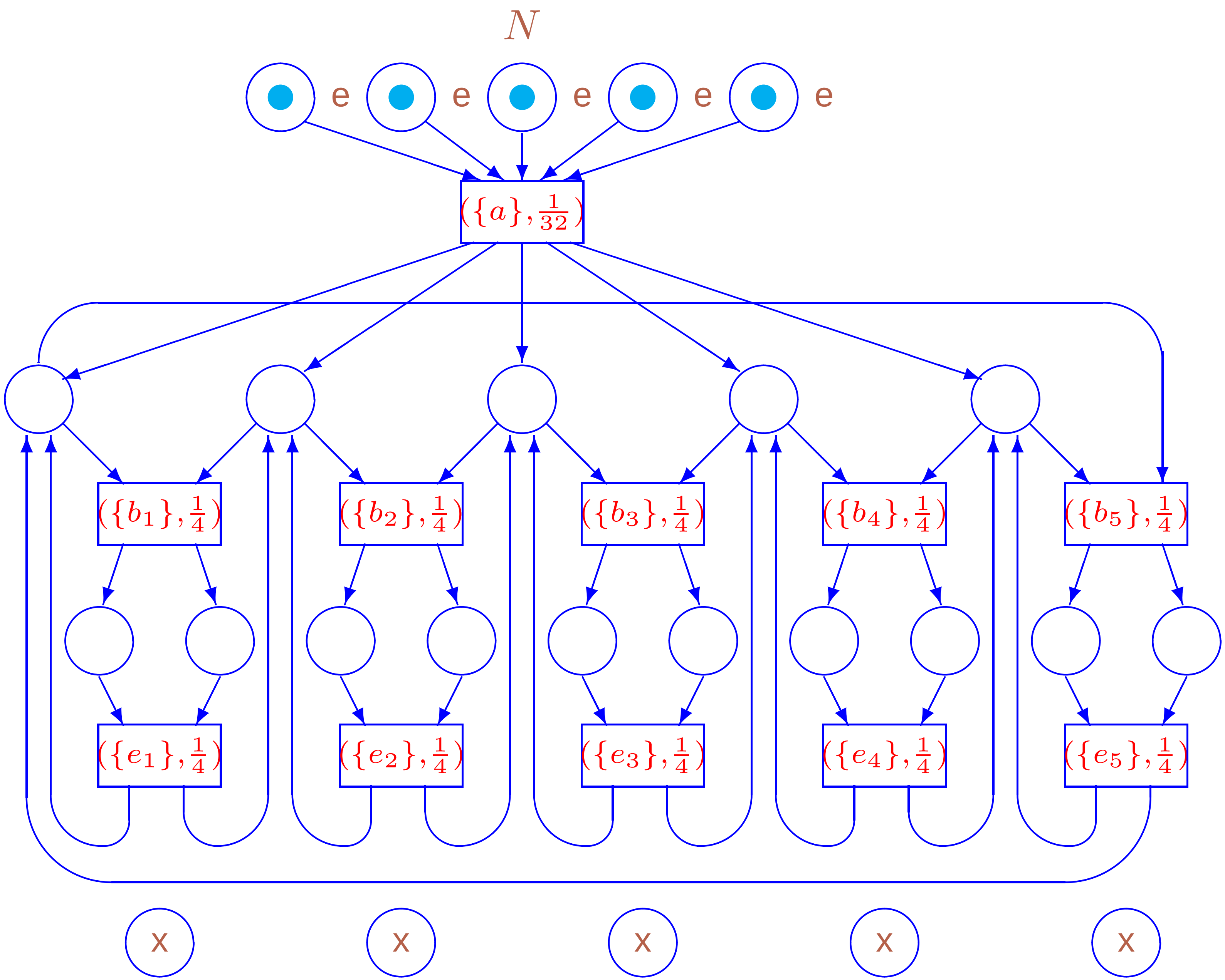
  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b_1\}, \frac{1}{4})$.

  The *steady-state probability of the beginning of eating of first philosopher* is

  $\psi_2^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_6^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_6) +$
  $\psi_7^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_7) =$
  $\frac{2}{11}\left(\frac{3}{20} + \frac{1}{20} + \frac{1}{20}\right) + \frac{1}{10}\left(\frac{3}{11} + \frac{1}{11}\right) + \frac{1}{10}\left(\frac{3}{11} + \frac{1}{11}\right) = \frac{13}{110}$.



The marked dts-boxes of the dining philosophers



The marked dts-box of the dining philosophers system

The abstract system

The static expression of the philosopher i $(1 \leq i \leq 4)$ is

$$F_i = [(\{x_i\}, \tfrac{1}{2}) * (((\{b, \widehat{y_i}\}, \tfrac{1}{2}); (\{e, \widehat{z_i}\}, \tfrac{1}{2}))[]((\{y_{i+1}\}, \tfrac{1}{2}); (\{z_{i+1}\}, \tfrac{1}{2}))) * \mathsf{Stop}].$$

The static expression of the philosopher 5 is

$$F_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b, \widehat{y_5}\}, \tfrac{1}{2}); (\{e, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2}))) * \mathsf{Stop}].$$

The static expression of the abstract dining philosophers system is

$$F = (F_1 \| F_2 \| F_3 \| F_4 \| F_5) \text{ sy } x_1 \text{ sy } x_2 \text{ sy } x_3 \text{ sy } x_4 \text{ sy } y_1 \text{ sy } y_2 \text{ sy } y_3 \text{ sy } y_4 \text{ sy } y_5 \text{ sy } z_1 \text{ sy } z_2 \text{ sy } z_3$$
$$\text{sy } z_4 \text{ sy } z_5 \text{ rs } x_1 \text{ rs } x_2 \text{ rs } x_3 \text{ rs } x_4 \text{ rs } y_1 \text{ rs } y_2 \text{ rs } y_3 \text{ rs } y_4 \text{ rs } y_5 \text{ rs } z_1 \text{ rs } z_2 \text{ rs } z_3 \text{ rs } z_4 \text{ rs } z_5.$$

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS^*(\overline{F})$ is similar to $TS^*(\overline{E})$.

$DTMC^*(\overline{F}) \simeq DTMC^*(\overline{E})$, thus, TPM and the steady-state PMF for $DTMC^*(\overline{F})$ and $DTMC^*(\overline{E})$ coincide.

Performance indices

The first performance index and the second group of the indices are the same for the standard and abstract systems.

The following performance index: non-personalized viewpoint to the philosophers.

- The beginning of eating of a philosopher $(\{b\}, \frac{1}{4})$ is only possible from the states $s_2, s_3, s_6, s_7, s_{10}, s_{11}$.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{1}{4})$.

  The *steady-state probability of the beginning of eating of a philosopher* is

$$\psi_2^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_3^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_3) +$$
$$\psi_6^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_6) + \psi_7^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_7) +$$
$$\psi_{10}^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_{10}) + \psi_{11}^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_{11}) =$$
$$\frac{2}{11}\left(\frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) +$$
$$\frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) +$$
$$\frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) = \frac{6}{11}.$$

The reduction of the abstract system

The static expression of the philosopher 1 is $F_1' = [(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5}); (\{e\}, \frac{1}{4})) * \mathsf{Stop}]$.

The static expression of the philosopher 2 is $F_2' = [(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5}); (\{e\}, \frac{1}{4})) * \mathsf{Stop}]$.

The static expression of the reduced abstract dining philosophers system is $F' = (F_1' \| F_2')\ \mathsf{sy}\ x\ \mathsf{rs}\ x$.

$DR(\overline{F'})$ consists of

$s_1' = [(([\overline{(\{x\}, \frac{1}{2})} * ((\{b\}, \frac{2}{5})_1; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \|$
$[\overline{(\{a, \hat{x}\}, \frac{1}{16})} * ((\{b\}, \frac{2}{5})_2; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_{\approx}$,

$s_2' = [(([(\{x\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{2}{5})_1}; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \|$
$[(\{a, \hat{x}\}, \frac{1}{16}) * (\overline{(\{b\}, \frac{2}{5})_2}; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_{\approx}$,

$s_3' = [(([(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5})_1; \overline{(\{e\}, \frac{1}{4})_1}) * \mathsf{Stop}] \|$
$[(\{a, \hat{x}\}, \frac{1}{16}) * (\overline{(\{b\}, \frac{2}{5})_2}; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_{\approx}$,

$s_4' = [(([(\{x\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{2}{5})_1}; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \|$
$[(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5})_2; \overline{(\{e\}, \frac{1}{4})_2}) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_{\approx}$,

$s_5' = [(([(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5})_1; \overline{(\{e\}, \frac{1}{4})_1}) * \mathsf{Stop}] \|$
$[(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5})_2; \overline{(\{e\}, \frac{1}{4})_2}) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_{\approx}$.

Interpretation of the states

s_1': the initial state,

s_2': the system is activated and no philosophers dine,

s_3', s_4': one philosopher dines,

s_5': two philosophers dine.

Consider $\mathcal{R} : \overline{F} \underline{\leftrightarrow}_{ss} \overline{F'}$ such that $(DR(\overline{F}) \cup DR(\overline{F'}))/_{\mathcal{R}} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$, where

$\mathcal{H}_1 = \{s_1, s_1'\}$ (the initial state),

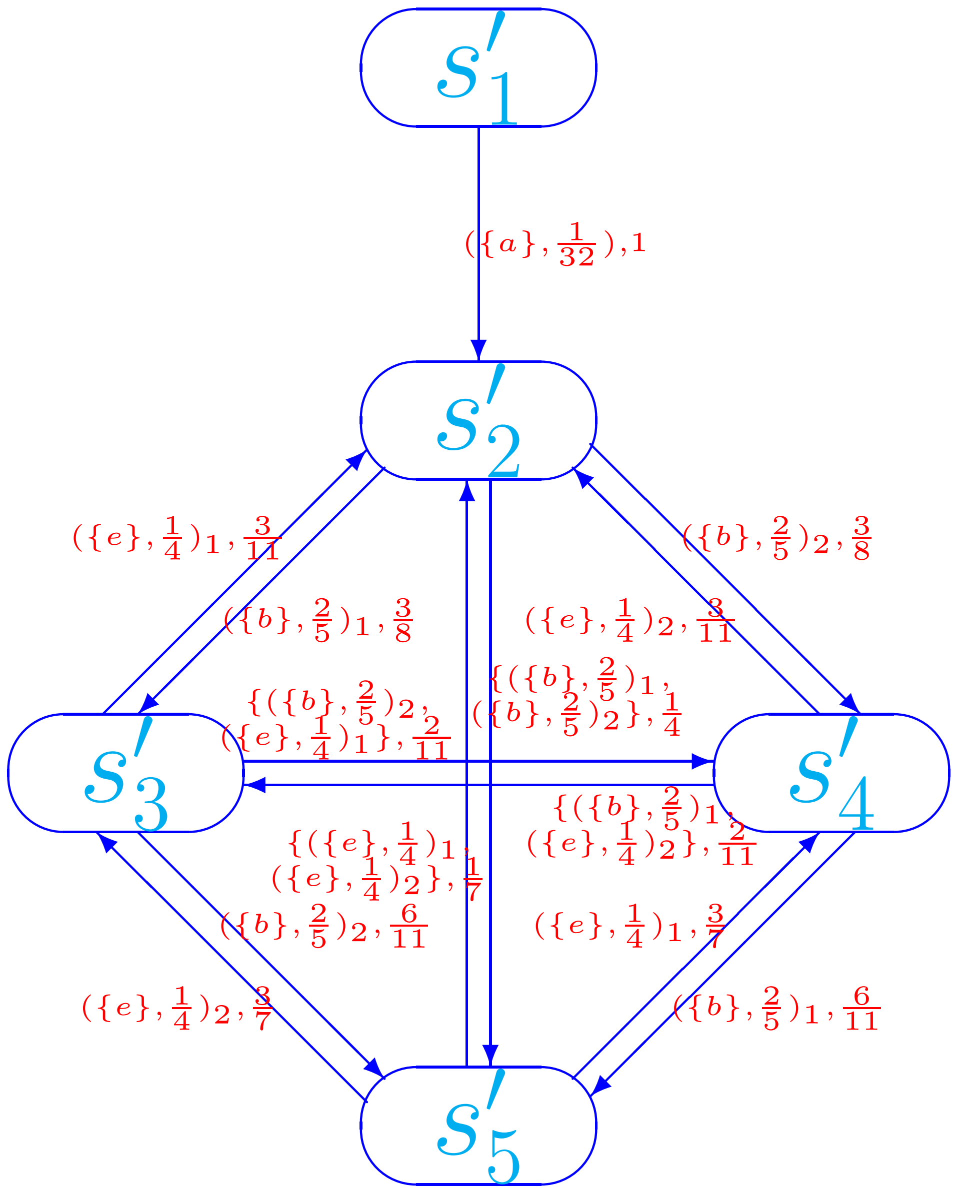
$\mathcal{H}_2 = \{s_2, s_2'\}$ (the system is activated and no philosophers dine),

$\mathcal{H}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}, s_3', s_4'\}$ (one philosopher dines),

$\mathcal{H}_4 = \{s_4, s_5, s_8, s_9, s_{12}, s_5'\}$ (two philosophers dine).

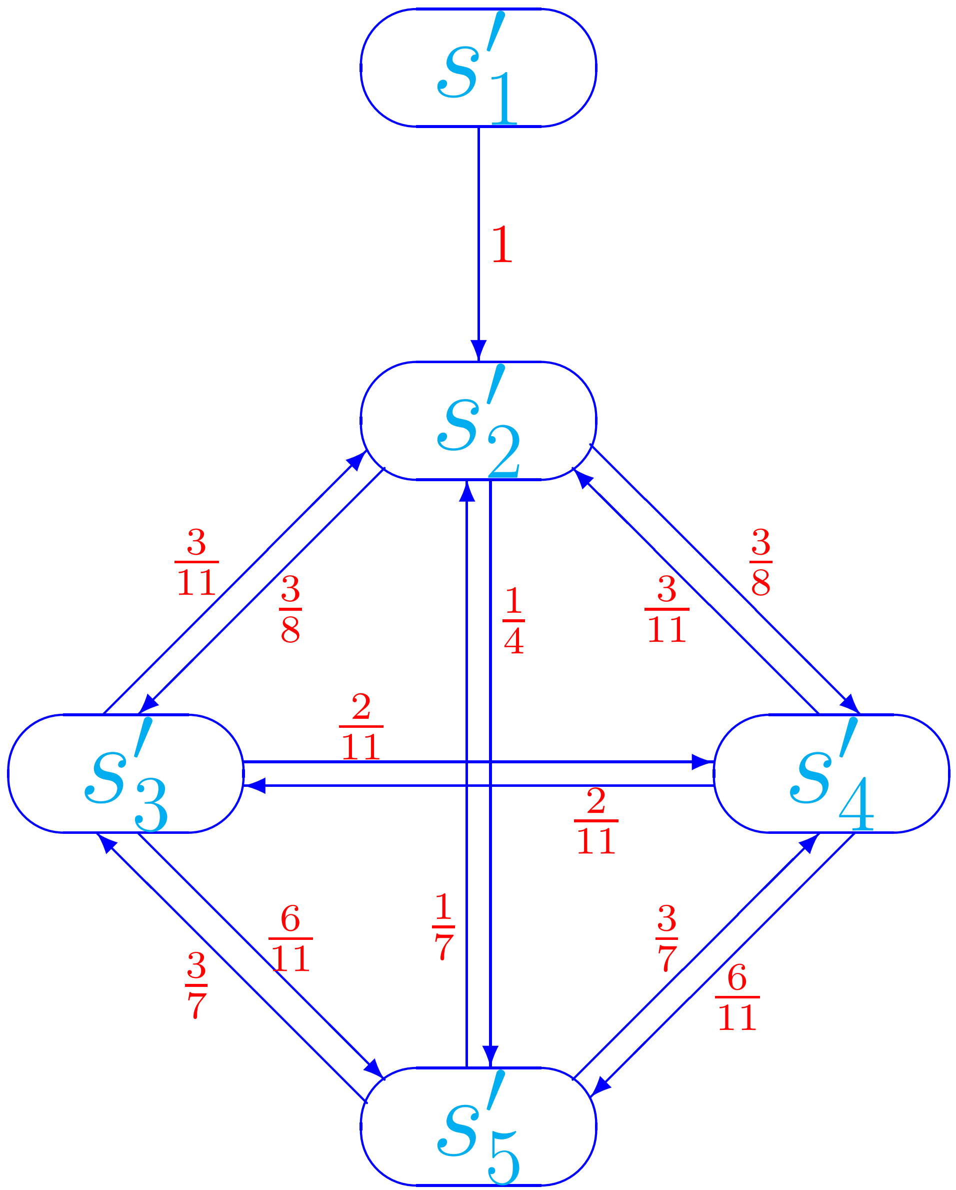
F' is a reduction of F w.r.t. $\underline{\leftrightarrow}_{ss}$.

$TS^*(\overline{F'})$



The transition system without empty loops of the reduced abstract dining philosophers system

$DTMC^*(\overline{F'})$



The underlying DTMC without empty loops of the reduced abstract dining philosophers system

The TPM for $DTMC^*(\overline{F'})$ is

$$\mathbf{P}'^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ 0 & \frac{3}{11} & 0 & \frac{2}{11} & \frac{6}{11} \\ 0 & \frac{3}{11} & \frac{2}{11} & 0 & \frac{6}{11} \\ 0 & \frac{1}{7} & \frac{3}{7} & \frac{3}{7} & 0 \end{pmatrix}.$$

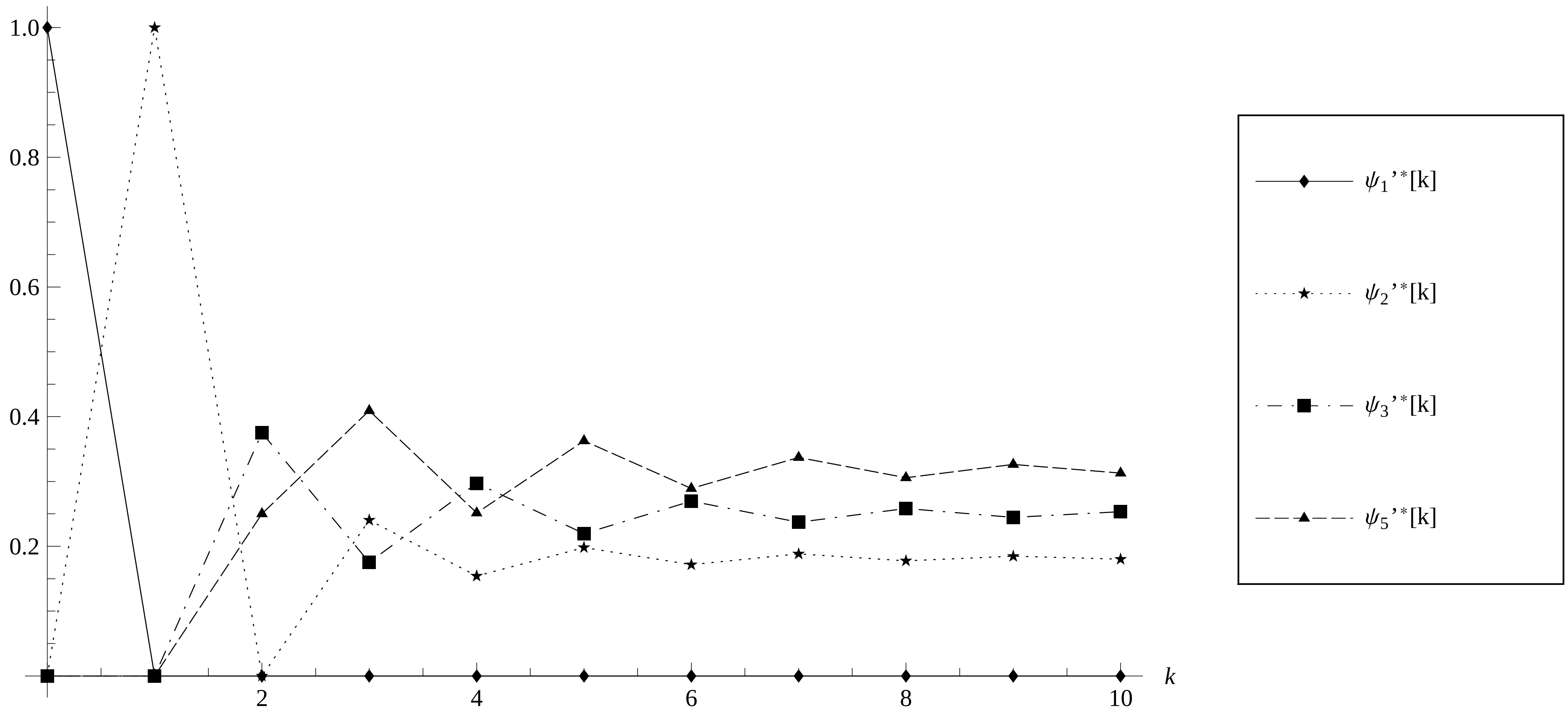
The steady-state PMF for $DTMC^*(\overline{F'})$ is

$$\psi'^* = \left(0, \frac{2}{11}, \frac{1}{4}, \frac{1}{4}, \frac{7}{22}\right).$$

Transient and steady-state probabilities of the reduced abstract dining philosophers system

k	0	1	2	3	4	5	6	7	8	9	10	∞
${\psi'_1}^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
${\psi'_2}^*[k]$	0	1	0	0.2403	0.1541	0.1981	0.1716	0.1884	0.1776	0.1846	0.1800	0.1818
${\psi'_3}^*[k]$	0	0	0.3750	0.1753	0.2973	0.2195	0.2697	0.2372	0.2583	0.2446	0.2535	0.2500
${\psi'_5}^*[k]$	0	0	0.2500	0.4091	0.2513	0.3628	0.2890	0.3371	0.3059	0.3261	0.3130	0.3182

We depict the probabilities for the states s'_1, s'_2, s'_3, s'_5 only, since the corresponding values coincide for s'_3, s'_4.



Transient probabilities alteration diagram of the reduced abstract dining philosophers system

Performance indices

- The average recurrence time in the state s_2', where all the forks are available, the *average system run-through*, is $\frac{1}{{\psi_2'}^*} = \frac{11}{2} = 5\frac{1}{2}$.

- Nobody eats in the state s_2'. The *fraction of time when no philosophers dine* is ${\psi_2'}^* = \frac{2}{11}$.

  Only one philosopher eats in the states s_3', s_4'. The *fraction of time when only one philosopher dines* is ${\psi_3'}^* + {\psi_4'}^* = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

  Two philosophers eat together in the state s_5'. The *fraction of time when two philosophers dine* is ${\psi_5'}^* = \frac{7}{22}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.

- The beginning of eating of a philosopher $(\{b\}, \frac{2}{5})$ is only possible from the states s_2', s_3', s_4'.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{2}{5})$.

  The *steady-state probability of the beginning of eating of a philosopher* is
  ${\psi_2'}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s_2') + {\psi_3'}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s_3') +$
  ${\psi_4'}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s_4') = \frac{2}{11}\left(\frac{3}{8} + \frac{3}{8} + \frac{1}{4}\right) + \frac{1}{4}\left(\frac{6}{11} + \frac{2}{11}\right) + \frac{1}{4}\left(\frac{6}{11} + \frac{2}{11}\right) = \frac{6}{11}$.

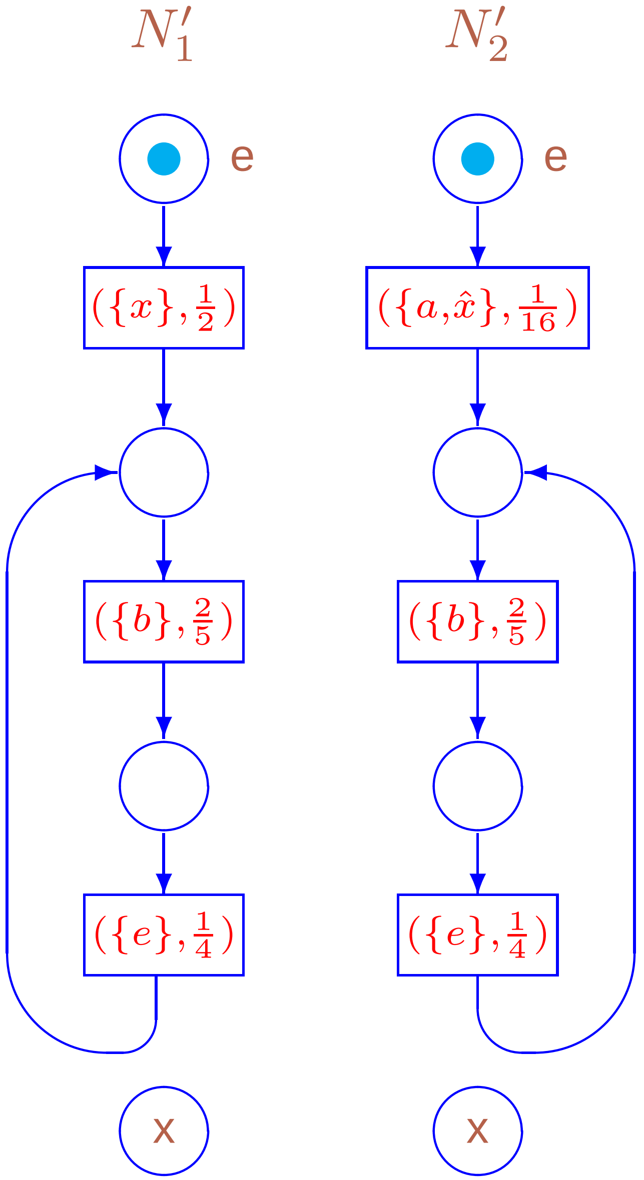
The performance indices are the same for the complete and the reduced abstract dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates proposition about steady-state probabilities.

The coincidence of the third performance index is by the theorem about derived step traces from steady states:

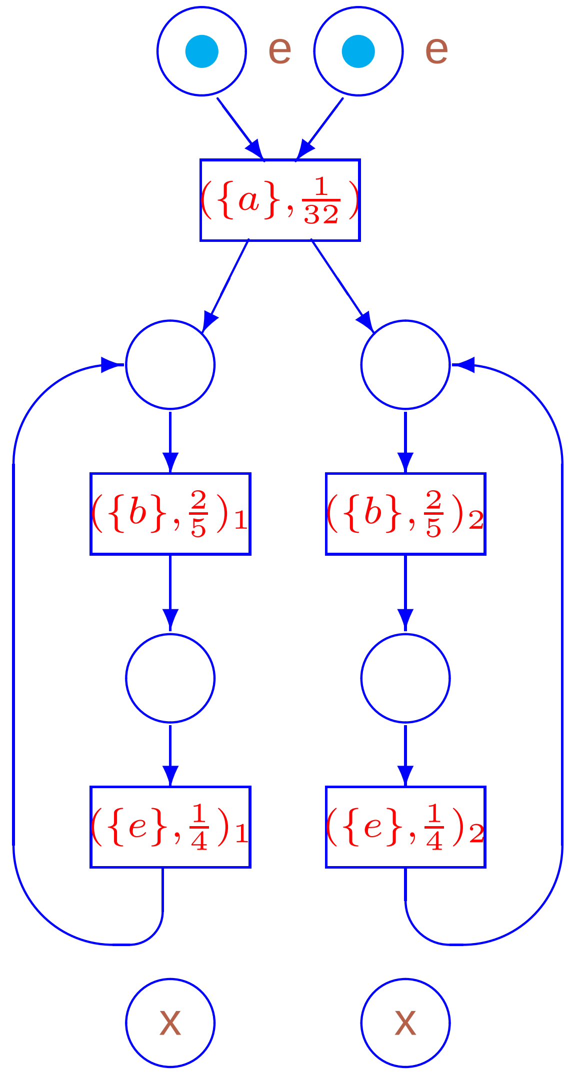
one should apply its result to the derived step traces $\{\{b\}\},\ \{\{b\},\{b\}\},\ \{\{b\},\{e\}\}$ of $\overline{F}$ and $\overline{F'}$,

and sum the left and right parts of the three resulting equalities.



The marked dts-boxes of the reduced abstract dining philosophers

N'



The marked dts-box of the reduced abstract dining philosophers system

The quotient of the abstract system

$DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}$, where

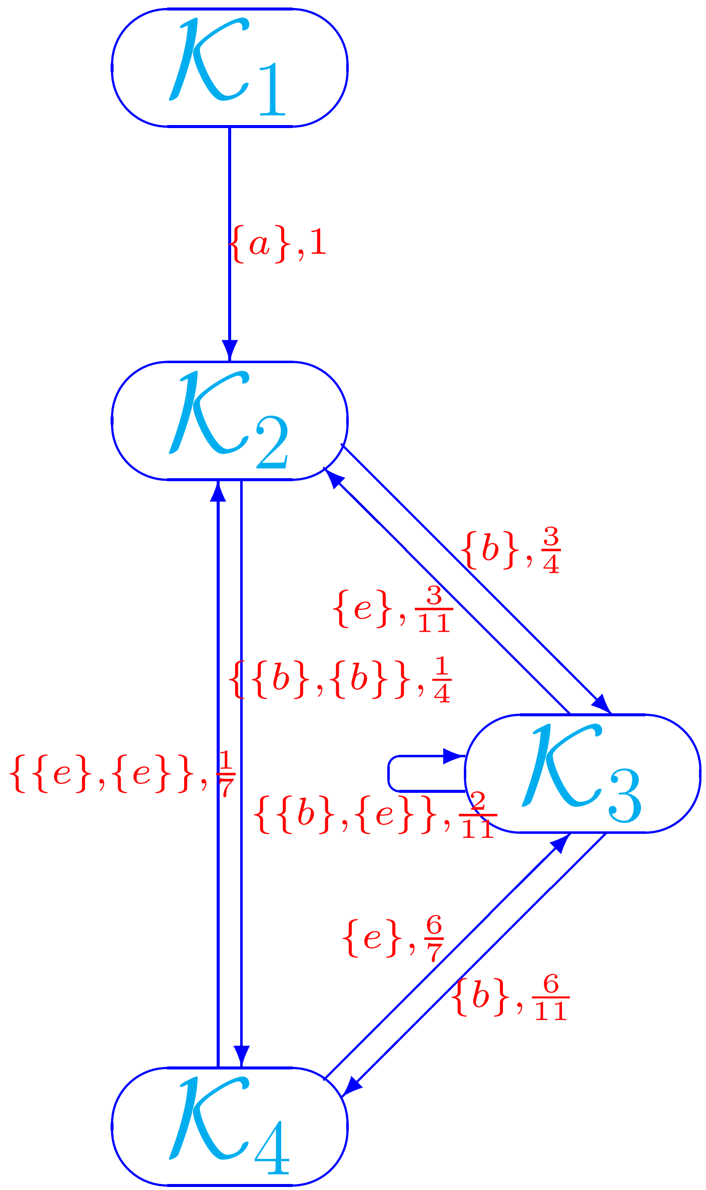
$\mathcal{K}_1 = \{s_1\}$ (the initial state),

$\mathcal{K}_2 = \{s_2\}$ (the system is activated and no philosophers dine),

$\mathcal{K}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}\}$ (one philosopher dines),

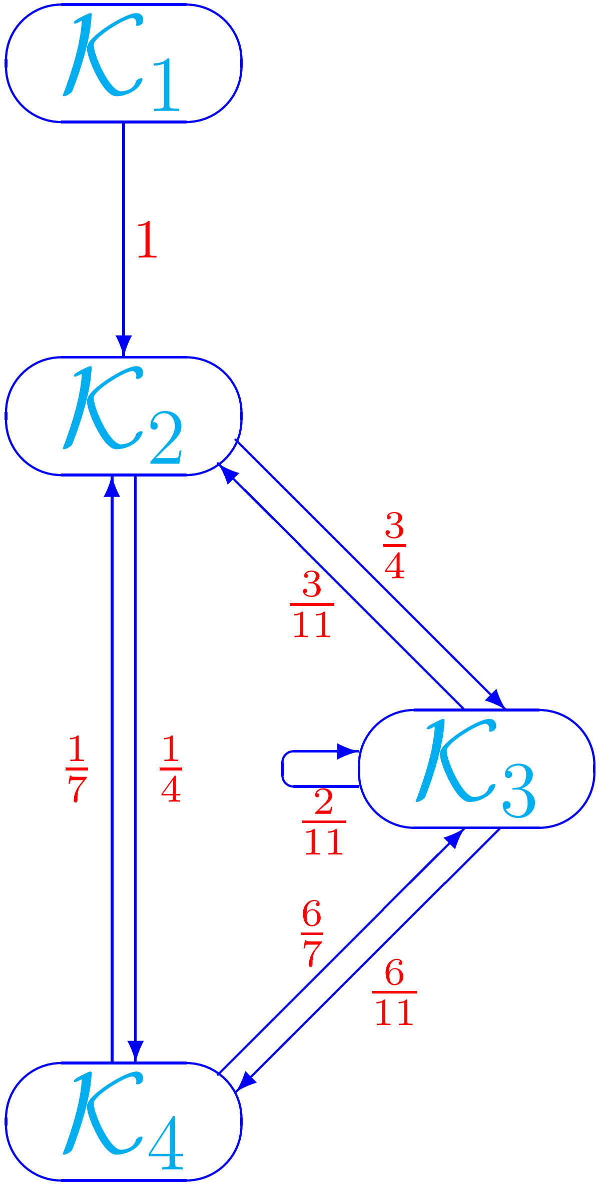
$\mathcal{K}_4 = \{s_4, s_5, s_8, s_9, s_{12}\}$ (two philosophers dine).

$TS^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



The quotient transition system without empty loops of the abstract dining philosophers system

$DTMC^{*}_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



The quotient underlying DTMC without empty loops of the abstract dining philosophers system

The TPM for $DTMC^{*}_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

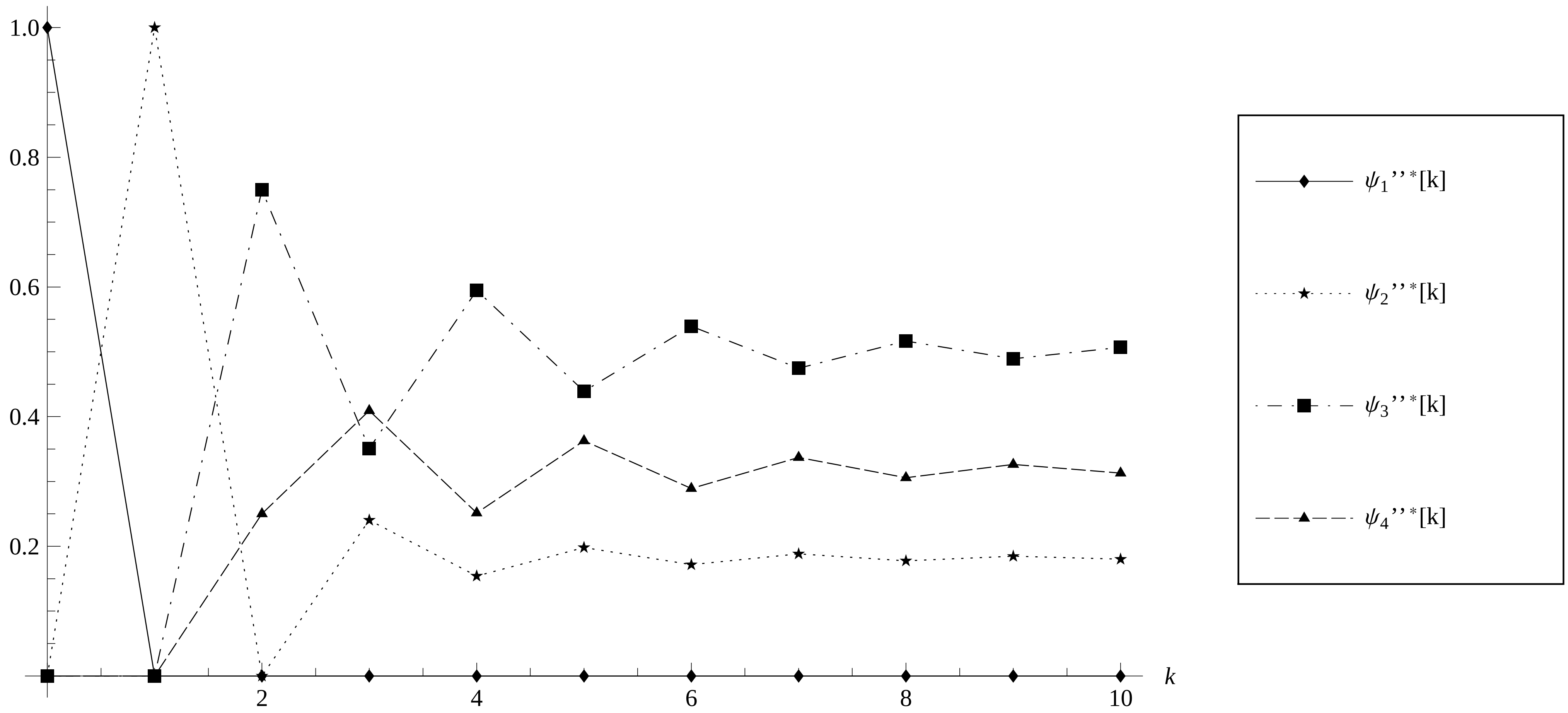
$$\mathbf{P}''^{*} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{11} & \frac{2}{11} & \frac{6}{11} \\ 0 & \frac{1}{7} & \frac{6}{7} & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^{*}_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

$$\psi''^{*} = \left(0, \frac{2}{11}, \frac{1}{2}, \frac{7}{22}\right).$$

Transient and steady-state probabilities of the quotient abstract dining philosophers system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1''^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2''^*[k]$	0	1	0	0.2403	0.1541	0.1981	0.1716	0.1884	0.1776	0.1846	0.1800	0.1818
$\psi_3''^*[k]$	0	0	0.7500	0.3506	0.5946	0.4391	0.5394	0.4745	0.5165	0.4893	0.5069	0.5000
$\psi_4''^*[k]$	0	0	0.2500	0.4091	0.2513	0.3628	0.2890	0.3371	0.3059	0.3261	0.3130	0.3182



Transient probabilities alteration diagram of the quotient abstract dining philosophers system

Performance indices

- The average recurrence time in the state $\mathcal{K}_2$, where all the forks are available, the *average system run-through*, is $\frac{1}{\psi_2''^*} = \frac{11}{2} = 5\frac{1}{2}$.

- Nobody eats in the state $\mathcal{K}_2$. The *fraction of time when no philosophers dine* is $\psi_2''^* = \frac{2}{11}$.

  Only one philosopher eats in the state $\mathcal{K}_3$. The *fraction of time when only one philosopher dines* is $\psi_3''^* = \frac{1}{2}$.

  Two philosophers eat together in the state $\mathcal{K}_4$. The *fraction of time when two philosophers dine* is $\psi_4''^* = \frac{7}{22}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.

- The beginning of eating of a philosopher $\{b\}$ is only possible from the states $\mathcal{K}_2, \mathcal{K}_3$.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of multiactions containing $\{b\}$.

  The *steady-state probability of the beginning of eating of a philosopher* is $\psi_2''^* \sum_{\{A,\mathcal{K} \mid \{b\}\in A,\ \mathcal{K}_2 \stackrel{A}{\twoheadrightarrow} \mathcal{K}\}} PM_A^*(\mathcal{K}_2, \mathcal{K}) + \psi_3''^* \sum_{\{A,\mathcal{K} \mid \{b\}\in A,\ \mathcal{K}_3 \stackrel{A}{\twoheadrightarrow} \mathcal{K}\}} PM_A^*(\mathcal{K}_3, \mathcal{K}) = \frac{2}{11}\left(\frac{3}{4} + \frac{1}{4}\right) + \frac{1}{2}\left(\frac{6}{11} + \frac{2}{11}\right) = \frac{6}{11}$.

The performance indices are the same for the complete and quotient abstract dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates proposition about steady-state probabilities.

The coincidence of the third performance index is by the theorem about derived step traces from steady states:

one should apply its result to the derived step traces $\{\{b\}\},\ \{\{b\},\{b\}\},\ \{\{b\},\{e\}\}$ of $\overline{F}$ and itself,

and sum the left and right parts of the three resulting equalities.

The generalized system

The static expression of the philosopher $i\ (1 \leq i \leq 4)$ is

$K_i = [(\{x_i\},\rho) * ((((\{b_i,\widehat{y_i}\},\rho);(\{e_i,\widehat{z_i}\},\rho))[]((\{y_{i+1}\},\rho);(\{z_{i+1}\},\rho))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is

$K_5 = [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\rho) * ((((\{b_5,\widehat{y_5}\},\rho);(\{e_5,\widehat{z_5}\},\rho))[]((\{y_1\},\rho);(\{z_1\},\rho))) * \mathsf{Stop}]$.

The static expression of the generalized dining philosophers system is

$K = (K_1\|K_2\|K_3\|K_4\|K_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

Interpretation of the states

$\tilde{s}_1$: the initial state,

$\tilde{s}_2$: the system is activated and no philosophers dine,

$\tilde{s}_3$: philosopher 1 dines,

$\tilde{s}_4$: philosophers 1 and 4 dine,

$\tilde{s}_5$: philosophers 1 and 3 dine,

$\tilde{s}_6$: philosopher 4 dines,

$\tilde{s}_7$: philosopher 3 dines,

$\tilde{s}_8$: philosophers 2 and 4 dine,

$\tilde{s}_9$: philosophers 3 and 5 dine,

$\tilde{s}_{10}$: philosopher 2 dines,

$\tilde{s}_{11}$: philosopher 5 dine,

$\tilde{s}_{12}$: philosophers 2 and 5 dine.

The TPM for $DTMC^*(\overline{K})$ is

$$\widetilde{\mathbf{P}}^* = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1-\rho^2}{5} & \frac{\rho^2}{5} & \frac{\rho^2}{5} & \frac{1-\rho^2}{5} & \frac{1-\rho^2}{5} & \frac{\rho^2}{5} & \frac{\rho^2}{5} & \frac{1-\rho^2}{5} & \frac{1-\rho^2}{5} & \frac{\rho^2}{5} \\
0 & \frac{1-\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} & \frac{1-\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1-\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & \frac{\rho^2}{3-\rho^2} & 0 & 0 \\
0 & \frac{1-\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & \frac{\rho^2}{3-\rho^2} & 0 \\
0 & \frac{\rho^2}{2-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 \\
0 & \frac{\rho^2}{2-\rho^2} & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 \\
0 & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & 0 & \frac{\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & \frac{\rho^2}{3-\rho^2} & \frac{1-\rho^2}{3-\rho^2} \\
0 & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & 0 & 0 & \frac{\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} \\
0 & \frac{\rho^2}{2-\rho^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & 0
\end{pmatrix}$$

The steady-state PMF for $DTMC^*(\overline{K})$ is $\tilde{\psi}^* =$

$$\left(0, \frac{1}{2(3-\rho^2)}, \frac{1}{10}, \frac{2-\rho^2}{10(3-\rho^2)}, \frac{2-\rho^2}{10(3-\rho^2)}, \frac{1}{10}, \frac{1}{10}, \frac{2-\rho^2}{10(3-\rho^2)}, \frac{2-\rho^2}{10(3-\rho^2)}, \frac{1}{10}, \frac{1}{10}, \frac{2-\rho^2}{10(3-\rho^2)}\right)$$

Performance indices

- The average recurrence time in the state s_2, where all the forks are available, the *average system run-through*, is $\frac{1}{\tilde{\psi}_2^*} = 2(3-\rho^2)$.

- Nobody eats in the state s_2. The *fraction of time when no philosophers dine* is $\tilde{\psi}_2^* = \frac{1}{2(3-\rho^2)}$.

  Only one philosopher eats in the states $s_3, s_6, s_7, s_{10}, s_{11}$. The *fraction of time when only one philosopher dines* is $\tilde{\psi}_3^* + \tilde{\psi}_6^* + \tilde{\psi}_7^* + \tilde{\psi}_{10}^* + \tilde{\psi}_{11}^* = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{1}{2}$.

  Two philosophers eat together in the states $s_4, s_5, s_8, s_9, s_{12}$. The *fraction of time when two philosophers dine* is $\tilde{\psi}_4^* + \tilde{\psi}_5^* + \tilde{\psi}_8^* + \tilde{\psi}_9^* + \tilde{\psi}_{12}^* =$
  $\frac{2-\rho^2}{10(3-\rho^2)} + \frac{2-\rho^2}{10(3-\rho^2)} + \frac{2-\rho^2}{10(3-\rho^2)} + \frac{2-\rho^2}{10(3-\rho^2)} + \frac{2-\rho^2}{10(3-\rho^2)} = \frac{2-\rho^2}{2(3-\rho^2)}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{2-\rho^2}{2(3-\rho^2)} \cdot \frac{2}{1} = \frac{2-\rho^2}{3-\rho^2}$.

- The beginning of eating of first philosopher $(\{b_1\}, \rho^2)$ is only possible from the states s_2, s_6, s_7.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b_1\}, \rho^2)$.

  The *steady-state probability of the beginning of eating of first philosopher* is
  $\tilde{\psi}_2^* \sum_{\{\Gamma|(\{b_1\},\rho^2)\in\Gamma\}} PT^*(\Gamma, s_2) + \tilde{\psi}_6^* \sum_{\{\Gamma|(\{b_1\},\rho^2)\in\Gamma\}} PT^*(\Gamma, s_6) +$
  $\tilde{\psi}_7^* \sum_{\{\Gamma|(\{b_1\},\rho^2)\in\Gamma\}} PT^*(\Gamma, s_7) =$
  $\frac{1}{2(3-\rho^2)}\left(\frac{1-\rho^2}{5} + \frac{\rho^2}{5} + \frac{\rho^2}{5}\right) + \frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2}\right) + \frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2}\right) = \frac{3+\rho^2}{10(3-\rho^2)}$.

The abstract generalized system

The static expression of the philosopher i $(1 \leq i \leq 4)$ is

$L_i = [(\{x_i\}, \rho) * (((\{b, \widehat{y_i}\}, \rho); (\{e, \widehat{z_i}\}, \rho))[]((\{y_{i+1}\}, \rho); (\{z_{i+1}\}, \rho))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is

$L_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \rho) * (((\{b, \widehat{y_5}\}, \rho); (\{e, \widehat{z_5}\}, \rho))[]((\{y_1\}, \rho); (\{z_1\}, \rho))) * \mathsf{Stop}]$.

The static expression of the abstract generalized dining philosophers system is

$L = (L_1 \| L_2 \| L_3 \| L_4 \| L_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

$DR(\overline{L})$ resembles $DR(\overline{K})$, and $TS^*(\overline{L})$ is similar to $TS^*(\overline{K})$.

$DTMC^*(\overline{L}) \simeq DTMC^*(\overline{K})$, thus, TPM and the steady-state PMF for $DTMC^*(\overline{L})$ and $DTMC^*(\overline{K})$ coincide.

Performance indices

The first performance index and the second group of the indices are the same for the generalized system and its abstract modification.

The following performance index: non-personalized viewpoint to the philosophers.

- The beginning of eating of a philosopher $(\{b\}, \rho^2)$ is only possible from the states $\tilde{s}_2, \tilde{s}_3, \tilde{s}_6, \tilde{s}_7, \tilde{s}_{10}, \tilde{s}_{11}$.

  The beginning of eating probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{b\}, \rho^2)$.

  The *steady-state probability of the beginning of eating of a philosopher* is

$$\tilde{\psi}_2^* \sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_2) + \tilde{\psi}_3^* \sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_3) + \tilde{\psi}_6^* \sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_6) + \tilde{\psi}_7^* \sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_7) + \tilde{\psi}_{10}^* \sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_{10}) + \tilde{\psi}_{11}^* \sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_{11}) =$$

$$\frac{1}{2(3-\rho^2)}\left(\frac{1-\rho^2}{5} + \frac{\rho^2}{5} + \frac{1-\rho^2}{5} + \frac{\rho^2}{5} + \frac{1-\rho^2}{5} + \frac{\rho^2}{5} + \frac{1-\rho^2}{5} + \frac{\rho^2}{5} + \frac{1-\rho^2}{5} + \frac{\rho^2}{5}\right) + \frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2} + \frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2}\right) + \frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2} + \frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2}\right) + \frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2} + \frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2}\right) + \frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2} + \frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2}\right) + \frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2} + \frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2}\right) = \frac{3}{2(3-\rho^2)}.$$

The reduction of the abstract generalized system

The static expression of the philosopher 1 is $L_1' = [(\{x\}, \rho) * ((\{b\}, \frac{2\rho^2}{1+\rho^2}); (\{e\}, \rho^2)) * \mathsf{Stop}]$.

The static expression of the philosopher 2 is $L_2' = [(\{a, \hat{x}\}, \rho^4) * ((\{b\}, \frac{2\rho^2}{1+\rho^2}); (\{e\}, \rho^2)) * \mathsf{Stop}]$.

The static expression of the reduced abstract generalized dining philosophers system is
$L' = (L_1' \| L_2')$ sy x rs x.

Consider $\mathcal{R} : \overline{L} \underline{\leftrightarrow}_{ss} \overline{L'}$ such that $(DR(\overline{L}) \cup DR(\overline{L'}))/_{\mathcal{R}} = \{\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2, \widetilde{\mathcal{H}}_3, \widetilde{\mathcal{H}}_4\}$, where

$\widetilde{\mathcal{H}}_1 = \{\tilde{s}_1, \tilde{s}'_1\}$ (the initial state),

$\widetilde{\mathcal{H}}_2 = \{\tilde{s}_2, \tilde{s}'_2\}$ (the system is activated and no philosophers dine),

$\widetilde{\mathcal{H}}_3 = \{\tilde{s}_3, \tilde{s}_6, \tilde{s}_7, \tilde{s}_{10}, \tilde{s}_{11}, \tilde{s}'_3, \tilde{s}'_4\}$ (one philosopher dines),

$\widetilde{\mathcal{H}}_4 = \{\tilde{s}_4, \tilde{s}_5, \tilde{s}_8, \tilde{s}_9, \tilde{s}_{12}, \tilde{s}'_5\}$ (two philosophers dine).

L' is a reduction of L w.r.t. $\underline{\leftrightarrow}_{ss}$.

The TPM for $DTMC^*(\overline{L'})$ is

$$\widetilde{\mathbf{P}}'^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\rho^2}{2} & \frac{1-\rho^2}{2} & \rho^2 \\ 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & \frac{2\rho^2}{3-\rho^2} & \frac{2(1-\rho^2)}{3-\rho^2} \\ 0 & \frac{1-\rho^2}{3-\rho^2} & \frac{2\rho^2}{3-\rho^2} & 0 & \frac{2(1-\rho^2)}{3-\rho^2} \\ 0 & \frac{\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^*(\overline{L'})$ is

$$\tilde{\psi}'^* = \left(0, \frac{1}{2(3-\rho^2)}, \frac{1}{4}, \frac{1}{4}, \frac{2-\rho^2}{2(3-\rho^2)}\right).$$

Performance indices

- The average recurrence time in the state $\tilde{s}'_2$, where all the forks are available, *average system run-through*, is $\frac{1}{\tilde{\psi}'^*_2} = 2(3-\rho^2)$.

- Nobody eats in the state $\tilde{s}'_2$. The *fraction of time when no philosophers dine* is $\tilde{\psi}'^*_2 = \frac{1}{2(3-\rho^2)}$.

  Only one philosopher eats in the states $\tilde{s}'_3, \tilde{s}'_4$. The *fraction of time when only one philosopher dines* is $\tilde{\psi}'^*_3 + \tilde{\psi}'^*_4 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

  Two philosophers eat together in the state $\tilde{s}'_5$. The *fraction of time when two philosophers dine* is $\tilde{\psi}'^*_5 = \frac{2-\rho^2}{2(3-\rho^2)}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{2-\rho^2}{2(3-\rho^2)} \cdot \frac{2}{1} = \frac{2-\rho^2}{3-\rho^2}$.

- The beginning of eating of a philosopher $(\{b\}, \frac{2\rho^2}{1+\rho^2})$ is only possible from the states $\tilde{s}'_2, \tilde{s}'_3, \tilde{s}'_4$.

  The beginning of eating probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{b\}, \frac{2\rho^2}{1+\rho^2})$.

  The *steady-state probability of the beginning of eating of a philosopher* is

  $$\tilde{\psi}'^*_2 \sum_{\{\Gamma|(\{b\},\frac{2\rho^2}{1+\rho^2})\in\Gamma\}} PT^*(\Gamma, \tilde{s}'_2) + \tilde{\psi}'^*_3 \sum_{\{\Gamma|(\{b\},\frac{2\rho^2}{1+\rho^2})\in\Gamma\}} PT^*(\Gamma, \tilde{s}'_3) + \tilde{\psi}'^*_4 \sum_{\{\Gamma|(\{b\},\frac{2\rho^2}{1+\rho^2})\in\Gamma\}} PT^*(\Gamma, \tilde{s}'_4) =$$

  $$\frac{1}{2(3-\rho^2)}\left(\frac{1-\rho^2}{2} + \frac{1-\rho^2}{2} + \rho^2\right) + \frac{1}{4}\left(\frac{2(1-\rho^2)}{3-\rho^2} + \frac{2\rho^2}{3-\rho^2}\right) + \frac{1}{4}\left(\frac{2(1-\rho^2)}{3-\rho^2} + \frac{2\rho^2}{3-\rho^2}\right) = \frac{3}{2(3-\rho^2)}.$$

The performance indices are the same for the complete and the reduced abstract generalized dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates proposition about steady-state probabilities.

The coincidence of the third performance index is by the theorem about derived step traces from steady states:

one should apply its result to the derived step traces $\{\{b\}\}, \{\{b\}, \{b\}\}, \{\{b\}, \{e\}\}$ of $\overline{L}$ and $\overline{L'}$,

and sum the left and right parts of the three resulting equalities.

The quotient of the abstract generalized system

$DR(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4\}$, where

$\widetilde{\mathcal{K}}_1 = \{\tilde{s}_1\}$ (the initial state),

$\widetilde{\mathcal{K}}_2 = \{\tilde{s}_2\}$ (the system is activated and no philosophers dine),

$\widetilde{\mathcal{K}}_3 = \{\tilde{s}_3, \tilde{s}_6, \tilde{s}_7, \tilde{s}_{10}, \tilde{s}_{11}\}$ (one philosopher dines),

$\widetilde{\mathcal{K}}_4 = \{\tilde{s}_4, \tilde{s}_5, \tilde{s}_8, \tilde{s}_9, \tilde{s}_{12}\}$ (two philosophers dine).

The TPM for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{L})$ is

$$\widetilde{\mathbf{P}}''^{*} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\rho^2 & \rho^2 \\ 0 & \frac{1-\rho^2}{3-\rho^2} & \frac{2\rho^2}{3-\rho^2} & \frac{2(1-\rho^2)}{3-\rho^2} \\ 0 & \frac{\rho^2}{2-\rho^2} & \frac{2(1-\rho^2)}{2-\rho^2} & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{L})$ is

$$\tilde{\psi}''^* = \left(0, \frac{1}{2(3-\rho^2)}, \frac{1}{2}, \frac{2-\rho^2}{2(3-\rho^2)}\right).$$

Performance indices

- The average recurrence time in the state $\widetilde{\mathcal{K}}_2$, where all the forks are available, the *average system run-through*, is $\frac{1}{\tilde{\psi}_2''^*} = 2(3-\rho^2)$.
- Nobody eats in the state $\widetilde{\mathcal{K}}_2$. The *fraction of time when no philosophers dine* is $\tilde{\psi}_2''^* = \frac{1}{2(3-\rho^2)}$.

  Only one philosopher eats in the state $\widetilde{\mathcal{K}}_3$. The *fraction of time when only one philosopher dines* is $\tilde{\psi}_3''^* = \frac{1}{2}$.

  Two philosophers eat together in the state $\widetilde{\mathcal{K}}_4$. The *fraction of time when two philosophers dine* is $\tilde{\psi}_4''^* = \frac{2-\rho^2}{2(3-\rho^2)}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{2-\rho^2}{2(3-\rho^2)} \cdot \frac{2}{1} = \frac{2-\rho^2}{3-\rho^2}$.

- The beginning of eating of a philosopher $\{b\}$ is only possible from the states $\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3$.

  The beginning of eating probability in each of the states is the sum of the execution probabilities for all multisets of multiactions containing $\{b\}$.

  The *steady-state probability of the beginning of eating of a philosopher* is
  $\tilde{\psi}_2''^{*} \sum_{\{A,\widetilde{\mathcal{K}} \mid \{b\}\in A,\ \widetilde{\mathcal{K}}_2 \overset{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}) + \tilde{\psi}_3''^{*} \sum_{\{A,\widetilde{\mathcal{K}} \mid \{b\}\in A,\ \widetilde{\mathcal{K}}_3 \overset{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}) = \frac{1}{2(3-\rho^2)}((1-\rho^2)+\rho^2) + \frac{1}{2}\left(\frac{2(1-\rho^2)}{3-\rho^2} + \frac{2\rho^2}{3-\rho^2}\right) = \frac{3}{2(3-\rho^2)}$.

The performance indices are the same for the complete and quotient abstract generalized dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates proposition about steady-state probabilities.

The coincidence of the third performance index is by the theorem about derived step traces from steady states:

one should apply its result to the derived step traces $\{\{b\}\},\ \{\{b\},\{b\}\},\ \{\{b\},\{e\}\}$ of $\overline{L}$ and itself,

and sum the left and right parts of the three resulting equalities.

Effect of quantitative changes of ρ to performance of the quotient abstract generalized dining philosophers system in its steady state

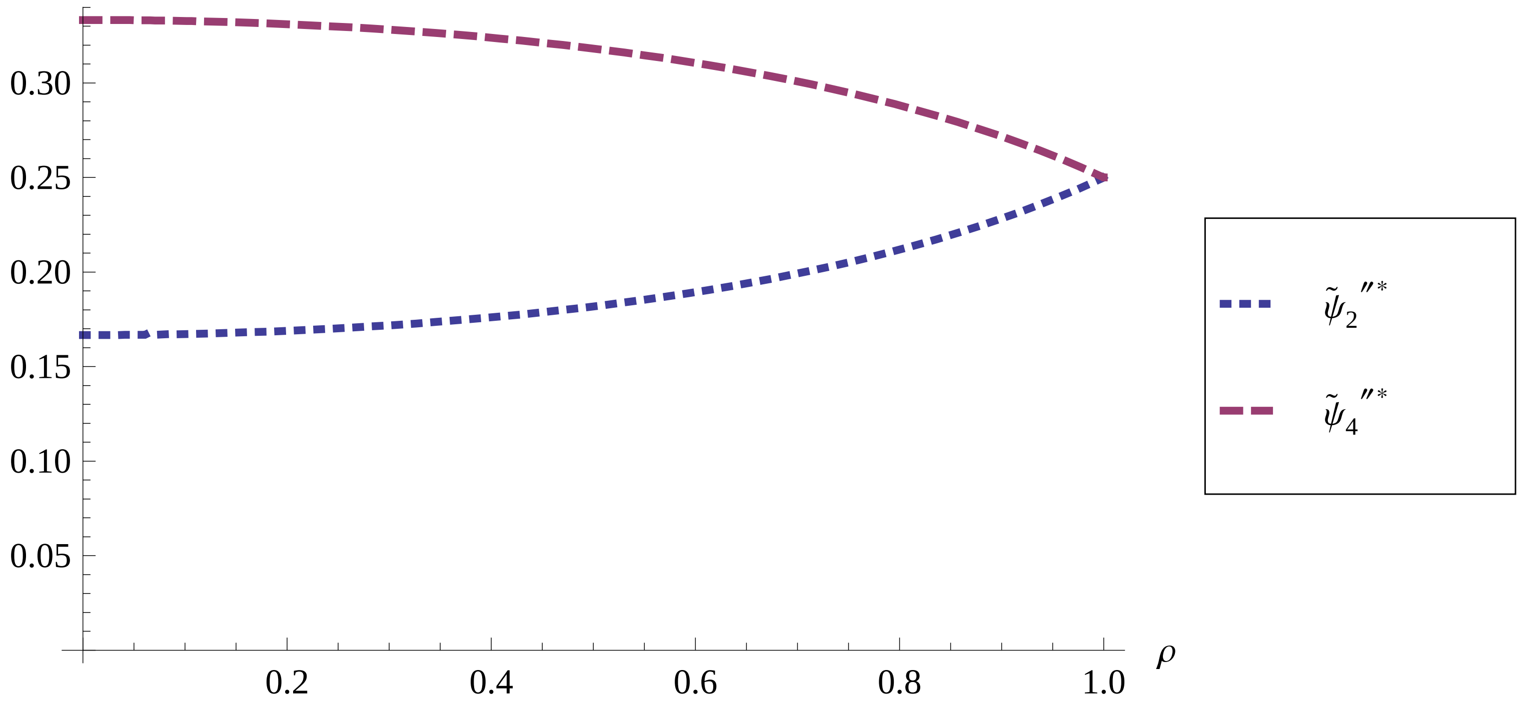
$\rho \in (0;1)$ is the probability of every multiaction of the system.

$\tilde{\psi}_1''^* = 0$ and $\tilde{\psi}_3''^* = \frac{1}{2}$ are constants, and they do not depend on ρ.

$\tilde{\psi}_2''^* = \frac{1}{2(3-\rho^2)}$ and $\tilde{\psi}_4''^* = \frac{2-\rho^2}{2(3-\rho^2)}$ depend on ρ.

$\tilde{\psi}_2''^* + \tilde{\psi}_4''^* = \frac{1}{2(3-\rho^2)} + \frac{2-\rho^2}{2(3-\rho^2)} = \frac{1}{2}$, hence, the sum of these steady-state probabilities does not depend on ρ.

Interpretation: the fraction of time when no or two philosophers dine coincides with that when only one philosopher dines, and both fractions are equal to $\frac{1}{2}$.



Steady-state probabilities $\tilde{\psi}_2''^*$ and $\tilde{\psi}_4''^*$ as functions of the parameter ρ

The diagrams in figure above are symmetric w.r.t. the probability $\frac{1}{4}$.

The more is value of ρ, the less is the difference $\tilde{\psi}_4''^* - \tilde{\psi}_2''^* = \frac{2-\rho^2}{2(3-\rho^2)} - \frac{1}{2(3-\rho^2)} = \frac{1-\rho^2}{2(3-\rho^2)}$.

The difference tends to $\frac{1}{6}$ when ρ approaches 0.

The difference tends to 0 when ρ approaches 1.

Interpretation: the difference between the fractions of time when two and when no philosophers dine.

More interesting value: $\tilde{\psi}_3''^{*} + \tilde{\psi}_4''^{*} - \tilde{\psi}_2''^{*} = \frac{1}{2} + \frac{2-\rho^2}{2(3-\rho^2)} - \frac{1}{2(3-\rho^2)} = \frac{2-\rho^2}{3-\rho^2}$.

The value tends to $\frac{2}{3}$ when ρ approaches 0.

The value tends to $\frac{1}{2}$ when ρ approaches 1.

Interpretation: the difference between the fractions of time when some (one or two) and when no philosophers dine.

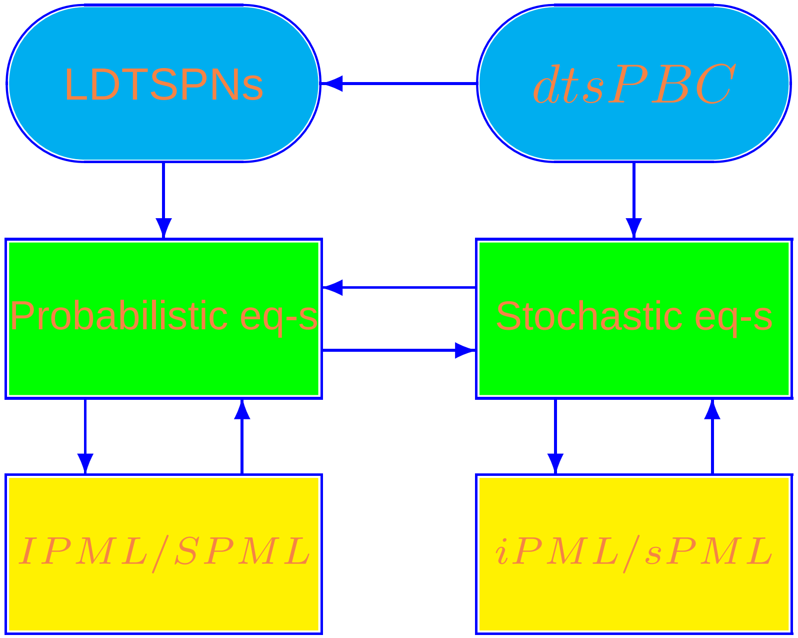
When ρ is closer to 0, more time is spent for eating and less time remains for thinking: *dining is preferred*.

When ρ is closer to 1, the situation is symmetric: *thinking is preferred*.

The influence of ρ to the performance indices presented before: similarly.

Overview and open questions

The results obtained



Stochastic formalisms and equivalences

- A discrete time stochastic extension $dtsPBC$ of finite PBC enriched with iteration.
- The step operational semantics based on labeled probabilistic transition systems.
- The denotational semantics in terms of a subclass of LDTSPNs.

- The stochastic algebraic equivalences which have natural net analogues on LDTSPNs.
- The transition systems and DTMCs reduction modulo stochastic equivalences.
- A logical characterization of stochastic bisimulation equivalences via probabilistic modal logics.
- An application of the equivalences to comparison of stationary behaviour.
- A preservation w.r.t. algebraic operations and the congruence relation.
- The case studies of performance analysis.

Further research

- Abstracting from silent activities in definitions of the equivalences.
- Introducing the immediate multiactions with zero delay.
- Extending the syntax with recursion operator.

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The slides can be downloaded from Internet:

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Thank you for your attention!