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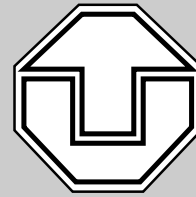
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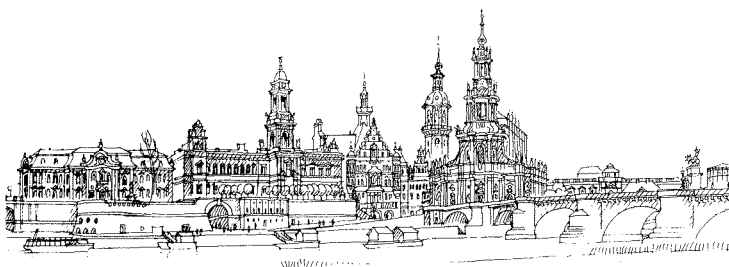## Technical Reports

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τ-**Equivalences and Refinement**
for Petri Nets Based Design

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Abstract

The paper is devoted to the investigation of behavioral equivalences of concurrent systems modeled by Petri nets with silent transitions. Basic τ-equivalences and back-forth τ-bisimulation equivalences known from the literature are supplemented by new ones, giving rise to complete set of equivalence notions in interleaving / true concurrency and linear / branching time semantics. Their interrelations are examined for the general class of nets as well as for their subclasses of nets without silent transitions and sequential nets (nets without concurrent transitions). In addition, the preservation of all the equivalence notions by refinements (allowing one to consider the systems to be modeled on a lower abstraction levels) is investigated.

Keywords: Petri nets with and without silent transitions, sequential nets, basic and back-forth τ-equivalences, refinement.

1 Introduction

The notion of equivalence is central in any theory of systems. It allows to compare systems taking into account particular aspects of their behavior.

Petri nets [16] became a popular formal model for design of concurrent and distributed systems. One of the main advantages of Petri nets is their ability for structural characterization of three fundamental features of concurrent computations: causality, nondeterminism and concurrency.

Silent transitions are transitions labeled by special *silent* action τ which represents an internal activity of a system to be modeled and it is invisible for external observer. It is well-known that Petri nets with silent transitions are more powerful than usual ones.

Equivalences which abstract of silent actions are called *τ-equivalences* (these are labeled by the symbol τ to distinguish them of relations not abstracting of silent actions). In recent years, a wide range of semantic equivalences was proposed in concurrency theory. Some of them were either directly defined or transferred from other formal models to Petri nets. The following *basic* notions of τ-equivalences are known from the literature.

- *τ-trace equivalences* (they respect only protocols of behavior of systems): interleaving ($\equiv_i^\tau$) [17], step ($\equiv_s^\tau$) [17], partial word ($\equiv_{pw}^\tau$) [25] and pomset ($\equiv_{pom}^\tau$) [18].

- *Usual τ-bisimulation equivalences* (they respect branching structure of behavior of systems): interleaving ($\underline{\leftrightarrow}_i^\tau$) [14], step ($\underline{\leftrightarrow}_s^\tau$) [17], partial word ($\underline{\leftrightarrow}_{pw}^\tau$) [24] and pomset ($\underline{\leftrightarrow}_{pom}^\tau$) [18].

- *ST-τ-bisimulation equivalences* (they respect the duration or maximality of events in behavior of systems): interleaving ($\underline{\leftrightarrow}_{iST}^\tau$) [24], partial word ($\underline{\leftrightarrow}_{pwST}^\tau$) [24] and pomset ($\underline{\leftrightarrow}_{pomST}^\tau$) [24].

- *History preserving τ-bisimulation equivalences* (they respect the "past" or "history" of behavior of systems): pomset ($\underline{\leftrightarrow}_{pomh}^\tau$) [9, 10].

- *History preserving ST-τ-bisimulation equivalences* (they respect the "history" and the duration or maximality of events in behavior of systems): pomset ($\underleftrightarrow{}^{\tau}_{pomhST}$) [9, 10].

- *Usual branching τ-bisimulation equivalences* (they respect branching structure of behavior of systems taking a special care for silent actions): interleaving ($\underleftrightarrow{}^{\tau}_{ibr}$) [12, 13].

- *History preserving branching τ-bisimulation equivalences* (they respect "history" and branching structure of behavior of systems taking a special care for silent actions): pomset ($\underleftrightarrow{}^{\tau}_{pomhbr}$) [9].

- *Isomorphism* ($\simeq$) (i.e. coincidence of systems up to renaming of their components).

Another type of equivalence notions called *back-forth* bisimulation equivalences are based on the idea that bisimulation relation do not only require systems to simulate each other behavior in the forward direction (as usually) but also when going back in history. They are closely connected with equivalences of logics with past modalities.

These equivalence notions were initially introduced in [15]. In the framework of transition systems without silent actions interleaving back-forth bisimulation equivalence ($\underleftrightarrow{}_{ibif}$) was defined and proved to merge with $\underleftrightarrow{}_{i}$. On transition systems with silent actions it was shown that back-forth variant ($\underleftrightarrow{}^{\tau}_{ibif}$) of interleaving τ-bisimulation equivalence coincide with $\underleftrightarrow{}^{\tau}_{ibr}$.

In [6, 7, 8], the new variants of step, partial word and pomset back-forth bisimulation equivalences were defined in the framework of prime event structures without silent actions.

In [19], the new idea of differentiating the kinds of back and forth simulations appeared (following this idea, it is possible, for example, to define step back pomset forth bisimulation equivalence). The set of all possible back-forth equivalence notions was proposed in interleaving, step, partial word and pomset semantics for prime event structures without silent actions. The new notion of τ-equivalence was proposed for event structures with silent actions: pomset back pomset forth ($\underleftrightarrow{}^{\tau}_{pombpomf}$) τ-bisimulation equivalence. It's coincidence with $\underleftrightarrow{}^{\tau}_{pomhbr}$ was proved.

To choose most appropriate behavioral viewpoint on systems to be modeled, it is very important to have a complete set of equivalence notions in all semantics and understand their interrelations. This branch of research is usually called *comparative concurrency semantics*. To clarify the nature of equivalences and evaluate how they respect internal activity and concurrency in systems to be modeled, it is actual to consider also correlation of these notions on nets without silent transitions and concurrency-free (sequential) ones. Treating equivalences for preservation by refinements allows one to decide which of them may be used for top-down design.

Working in the framework of Petri nets with silent transitions, in this paper we continue the research of [20, 21, 22] and extend the set of basic notions of τ-equivalences by interleaving ST-branching τ-bisimulation one ($\underleftrightarrow{}^{\tau}_{iSTbr}$), pomset history preserving ST-branching τ-bisimulation one ($\underleftrightarrow{}^{\tau}_{pomhSTbr}$) and multi event structure conflict preserving one ($\equiv^{\tau}_{mes}$). Let us note that an idea to introduce $\underleftrightarrow{}^{\tau}_{pomhSTbr}$ appeared initially in [19] on the model of event structures. We complete back-forth τ-equivalences from [19] by 6 new notions: interleaving back step forth ($\underleftrightarrow{}^{\tau}_{ibsf}$), interleaving back partial word forth ($\underleftrightarrow{}^{\tau}_{ibpwf}$), interleaving back pomset forth ($\underleftrightarrow{}^{\tau}_{ibpomf}$), step back step forth ($\underleftrightarrow{}^{\tau}_{sbsf}$), step back partial word forth ($\underleftrightarrow{}^{\tau}_{sbpwf}$) and step back pomset forth ($\underleftrightarrow{}^{\tau}_{sbpomf}$) τ-bisimulation equivalences. We compare all back-forth τ-equivalences with the set of basic behavioral relations.

We also investigate the interrelations of all the considered τ-equivalences with equivalences which do not abstract of silent actions.

In [5], SM-refinement operator for Petri nets was proposed, which "replaces" their transitions by SM-nets, a special subclass of state machine nets. We treat all the considered τ-equivalence notions for preservation by SM-refinements. We show that $\underleftrightarrow{}^{\tau}_{iSTbr}$, $\underleftrightarrow{}^{\tau}_{pomhSTbr}$ and $\equiv^{\tau}_{mes}$, i.e. all the new basic equivalences considered in this paper, are preserved by SM-refinements. Thus, we have branching and conflict preserving equivalences which may be used for multilevel design. In the literature, a stability w.r.t. SM-refinements was proved only for $\underleftrightarrow{}^{\tau}_{pomhST}$ in [5] and for $\underleftrightarrow{}^{\tau}_{iST}$ in [10]. The preservation result for other ST-τ-bisimulation equivalences was proved in [24], but it was done on event structures and an other refinement operator was used. The preservation of trace τ-equivalences was not established before. Thus, our results for $\underleftrightarrow{}^{\tau}_{pwST}$, $\underleftrightarrow{}^{\tau}_{pomST}$, $\equiv^{\tau}_{pw}$ and $\equiv^{\tau}_{pom}$ are also new.

In addition, we investigate the interrelations of all the τ-equivalence notions on nets without silent transitions and sequential nets. We prove that on nets without silent transitions τ-equivalences coincide with equivalence notions which do not abstract of silent actions. We demonstrate that on sequential nets interleaving and pomset τ-equivalences are merged, and back-forth τ-equivalences coincide with forth τ-equivalence relations.

The rest of the paper is organized as follows. Basic definitions are introduced in Section 2. In Section 3, we propose basic τ-equivalences and investigate their interrelations. In Section 4, back-forth τ-bisimulation equivalences are defined and compared with basic τ-equivalence notions. All the considered τ-equivalences are compared with ones which do not abstract of silent actions in Section 5. In Section 6, we establish which τ-equivalence relations are preserved by SM-refinements. Section 7 is devoted to comparison of the τ-equivalences

on nets without silent transitions and sequential nets. Concluding Section 8 contains a review of the main results obtained and some directions of further research.

2 Basic definitions

In this section, we present some basic definitions used further.

2.1 Multisets

A multiset is an extension of a set notion allowing an existence of several identical elements in the latter.

Definition 2.1 *Let X be some set. A finite multiset M over X is a mapping $M : X \to I\!\!N$ s.t. $|\{x \in X \mid M(x) > 0\}| < \infty$.*

$\mathcal{M}(X)$ denotes the *set of all finite multisets* over X. When $\forall x \in X \; M(x) \leq 1$, M is a proper set. The *cardinality* of a multiset M is defined in such a way: $|M| = \sum_{x \in X} M(x)$. We write $x \in M$ if $M(x) > 0$ and $M \subseteq M'$, if $\forall x \in X \; M(x) \leq M'(x)$. We define $(M + M')(x) = M(x) + M'(x)$ and $(M - M')(x) = \max\{0, M(x) - M'(x)\}$.

2.2 Labeled nets

A labeled net is a Petri net s.t. its transitions may be "labeled" by action (a sort of activity) names.

Let $Act = \{a, b, \ldots\}$ be a set of *action names* or *labels*. The symbol $\tau \notin Act$ denotes a special *silent* action which represents internal activity of system to be modeled and invisible to external observer. We denote $Act_\tau = Act \cup \{\tau\}$.

Definition 2.2 *A labeled net is a quadruple $N = \langle P_N, T_N, F_N, l_N \rangle$, where:*

- $P_N = \{p, q, \ldots\}$ *is a set of* places;

- $T_N = \{t, u, \ldots\}$ *is a set of* transitions;

- $F_N : (P_N \times T_N) \cup (T_N \times P_N) \to I\!\!N$ *is the* flow relation *with weights*;

- $l_N : T_N \to Act_\tau$ *is a* labeling *of transitions with action names.*

Given labeled nets $N = \langle P_N, T_N, F_N, l_N \rangle$ and $N' = \langle P_{N'}, T_{N'}, F_{N'}, l_{N'} \rangle$. A mapping $\beta : P_N \cup T_N \to P_{N'} \cup T_{N'}$ is an *isomorphism* between N and N', denoted by $\beta : N \simeq N'$, if:

1. β is a bijection s.t. $\beta(P_N) = P_{N'}$ and $\beta(T_N) = T_{N'}$;

2. $\forall p \in P_N \; \forall t \in T_N \; F_N(p, t) = F_{N'}(\beta(p), \beta(t))$ and $F_N(t, p) = F_{N'}(\beta(t), \beta(p))$;

3. $\forall t \in T_N \; l_N(t) = l_{N'}(\beta(t))$.

Labeled nets N and N' are *isomorphic*, denoted by $N \simeq N'$, if $\exists \beta : N \simeq N'$.

Given a labeled net N and some transition $t \in T_N$, the *precondition* and *postcondition* of t, denoted by $^\bullet t$ and $t^\bullet$ respectively, are the multisets defined in such a way: $(^\bullet t)(p) = F_N(p, t)$ and $(t^\bullet)(p) = F_N(t, p)$. Analogous definitions are introduced for places: $(^\bullet p)(t) = F_N(t, p)$ and $(p^\bullet)(t) = F_N(p, t)$. Let $^\circ N = \{p \in P_N \mid {}^\bullet p = \emptyset\}$ be a set of *initial (input)* places of N and $N^\circ = \{p \in P_N \mid p^\bullet = \emptyset\}$ be a set of *final (output)* places of N.

A labeled net N is *acyclic*, if there exist no transitions $t_0, \ldots, t_n \in T_N$ s.t. $t_{i-1}^\bullet \cap {}^\bullet t_i \neq \emptyset$ ($1 \leq i \leq n$) and $t_0 = t_n$. A labeled net N is *ordinary* if $\forall t \in T_N \; {}^\bullet t$ and $t^\bullet$ are proper sets (not multisets).

Let $N = \langle P_N, T_N, F_N, l_N \rangle$ be acyclic ordinary labeled net and $x, y \in P_N \cup T_N$. Let us introduce the following notions.

- $x \prec_N y \iff x F_N^+ y$, where F_N^+ is a transitive closure of F_N (*strict causal dependence* relation);

- $x \preceq_N y \iff (x \prec_N y) \vee (x = y)$ (a relation of *causal dependence*);

- $x \#_N y \iff \exists t, u \in T_N \; (t \neq u, \; {}^\bullet t \cap {}^\bullet u \neq \emptyset, \; t \preceq_N x, \; u \preceq_N y)$ (a relation of *conflict*);

- $\downarrow_N x = \{y \in P_N \cup T_N \mid y \prec_N x\}$ (the set of *strict predecessors* of x).

A set $T \subseteq T_N$ is *left-closed* in N, if $\forall t \in T \; (\downarrow_N t) \cap T_N \subseteq T$.

2.3 Marked nets

A marked net is a labeled net having active elements called tokens in its places. Such the places are considered to be "marked". Thus, a behavior of a marked net can be considered, in accordance to the special rules of the "token game."

A *marking* of a labeled net N is a multiset $M \in \mathcal{M}(P_N)$.

Definition 2.3 *A* (marked) net *is a tuple* $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$, *where* $\langle P_N, T_N, F_N, l_N \rangle$ *is a labeled net and* $M_N \in \mathcal{M}(P_N)$ *is the* initial *marking.*

Given nets $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$ and $N' = \langle P_{N'}, T_{N'}, F_{N'}, l_{N'}, M_{N'} \rangle$. A mapping $\beta : P_N \cup T_N \to P_{N'} \cup T_{N'}$ is an *isomorphism* between N and N', denoted by $\beta : N \simeq N'$, if:

1. $\beta : \langle P_N, T_N, F_N, l_N \rangle \simeq \langle P_{N'}, T_{N'}, F_{N'}, l_{N'} \rangle$;

2. $\forall p \in P_N \ M_N(p) = M_{N'}(\beta(p))$.

Nets N and N' are *isomorphic*, denoted by $N \simeq N'$, if $\exists \beta : N \simeq N'$.

Let $M \in \mathcal{M}(P_N)$ be a marking of a net N. A transition $t \in T_N$ is *fireable* in M, if $^\bullet t \subseteq M$. If t is fireable in M, its firing yields a new marking $\widetilde{M} = M - {}^\bullet t + t^\bullet$, denoted by $M \xrightarrow{t} \widetilde{M}$. A marking M of a net N is *reachable*, if $M = M_N$ or there exists a reachable marking $\widehat{M}$ of N s.t. $\widehat{M} \xrightarrow{t} M$ for some $t \in T_N$. $Mark(N)$ denotes a *set of all reachable* markings of a net N.

2.4 Partially ordered sets

A partially ordered set (poset) is a special formalism used for a semantic description of concurrent systems. Posets allow one to specify causal dependencies of events of a modeled system. Concurrency is interpreted as causal independence.

Definition 2.4 *A* labeled partially ordered set (lposet) *is a triple* $\rho = \langle X, \prec, l \rangle$, *where:*

- $X = \{x, y, \dots\}$ *is some set;*

- $\prec \subseteq X \times X$ *is a strict partial order (irreflexive transitive relation) over* X, *a* causal dependence *relation;*

- $l : X \to Act_\tau$ *is a* labeling *function.*

Let $\rho = \langle X, \prec, l \rangle$ be lposet and $x \in X, Y \subseteq X$. Then $\downarrow x = \{y \in X \mid y \prec x\}$ is a set of *strict predecessors* of x. A *restriction* of ρ to the set Y is defined as follows: $\rho|_Y = \langle Y, \prec \cap (Y \times Y), l|_Y \rangle$.

Let $\rho = \langle X, \prec, l \rangle$ and $\rho' = \langle X', \prec', l' \rangle$ be lposets.

A mapping $\beta : X \to X'$ is a *label-preserving bijection* between ρ and ρ', denoted by $\beta : \rho \asymp \rho'$, if:

1. β is a bijection;

2. $\forall x \in X \ l(x) = l'(\beta(x))$.

We write $\rho \asymp \rho'$, if $\exists \beta : \rho \asymp \rho'$.

A mapping $\beta : X \to X'$ is a *homomorphism* between ρ and ρ', denoted by $\beta : \rho \sqsubseteq \rho'$, if:

1. $\beta : \rho \asymp \rho'$;

2. $\forall x, y \in X \ x \prec y \ \Rightarrow \ \beta(x) \prec' \beta(y)$.

We write $\rho \sqsubseteq \rho'$, if $\exists \beta : \rho \sqsubseteq \rho'$.

A mapping $\beta : X \to X'$ is an *isomorphism* between ρ and ρ', denoted by $\beta : \rho \simeq \rho'$, if $\beta : \rho \sqsubseteq \rho'$ and $\beta^{-1} : \rho' \sqsubseteq \rho$. Two lposets ρ and ρ' are *isomorphic*, denoted by $\rho \simeq \rho'$, if $\exists \beta : \rho \simeq \rho'$.

Definition 2.5 Partially ordered multiset (pomset) *is an isomorphism class of lposets.*

2.5 Event structures

An event structure is an extension of a pomset notion which allows one to specify conflicts between events, i.e. the situations when an occurrence of one event excludes that of another.

Definition 2.6 *A* labeled event structure (LES) *is a quadruple* $\xi = \langle X, \prec, \#, l \rangle$, *where:*

- $X = \{x, y, \ldots\}$ *is a set of* events;

- $\prec \subseteq X \times X$ *is a strict partial order over* X, *a* causal dependence *relation, which satisfies to the principle of* finite causes: $\forall x \in X \mid \downarrow x \mid < \infty$;

- $\# \subseteq X \times X$ *is an irreflexive symmetrical* conflict relation, *which satisfies to the principle of* conflict heredity: $\forall x, y, z \in X \ x \# y \prec z \ \Rightarrow \ x \# z$;

- $l : X \to Act_\tau$ *is a* labeling *function.*

Let $\xi = \langle X, \prec, \#, l \rangle$ be LES and $Y \subseteq X$. A *restriction* of ξ to the set Y is defined as follows: $\xi|_Y = \langle Y, \prec \cap (Y \times Y), \# \cap (Y \times Y), l|_Y \rangle$.

Let $\xi = \langle X, \prec, \#, l \rangle$ and $\xi' = \langle X', \prec', \#', l' \rangle$ be LES's. A mapping $\beta : X \to X'$ is an *isomorphism* between ξ and ξ', denoted by $\beta : \xi \simeq \xi'$, if:

1. $\beta : \langle X, \prec, l \rangle \simeq \langle X', \prec', l' \rangle$;

2. $\forall x, y \in X \ x \# y \ \Leftrightarrow \ \beta(x) \#' \beta(y)$.

Two LES's ξ and ξ' are *isomorphic*, denoted by $\xi \simeq \xi'$, if $\exists \beta : \xi \simeq \xi'$.

Definition 2.7 *A* multi-event structure (MES) *is an isomorphism class of LES's.*

2.6 Processes

A process [4] may be considered as a formalism describing a particular computation of a modeled system. Usually, processes are deterministic, since in a computation no two events may be involved into a conflict (all of them will occur).

Definition 2.8 *A* causal net *is an acyclic ordinary labeled net* $C = \langle P_C, T_C, F_C, l_C \rangle$, *s.t.:*

1. $\forall r \in P_C \ |{}^\bullet r| \leq 1$ *and* $|r^\bullet| \leq 1$, *i.e. places are unbranched;*

2. $\forall x \in P_C \cap T_C \mid \downarrow_C x \mid < \infty$, *i.e. a set of causes is finite.*

Let us note that on the basis of any causal net $C = \langle P_C, T_C, F_C, l_C \rangle$ one can define lposet $\rho_C = \langle T_C, \prec_N \cap (T_C \times T_C), l_C \rangle$.

The fundamental property of causal nets is [2]: if C is a causal net, then there exists a sequence of transition firings ${}^\circ C = L_0 \overset{v_1}{\to} \cdots \overset{v_n}{\to} L_n = C^\circ$ s.t. $L_i \subseteq P_C$ ($0 \leq i \leq n$), $P_C = \cup_{i=0}^n L_i$ and $T_C = \{v_1, \ldots, v_n\}$. Such a sequence is called a *full execution* of C.

Definition 2.9 *Given a net* N *and a causal net* C. *A mapping* $\varphi : P_C \cup T_C \to P_N \cup T_N$ *is an* embedding *of* C *into* N, *denoted by* $\varphi : C \to N$, *if:*

1. $\varphi(P_C) \in \mathcal{M}(P_N)$ *and* $\varphi(T_C) \in \mathcal{M}(T_N)$, *i.e. sorts are preserved;*

2. $\forall v \in T_C \ {}^\bullet \varphi(v) = \varphi({}^\bullet v)$ *and* $\varphi(v)^\bullet = \varphi(v^\bullet)$, *i.e. flow relation is respected;*

3. $\forall v \in T_C \ l_C(v) = l_N(\varphi(v))$, *i.e. labeling is preserved.*

Since embeddings respect the flow relation, if ${}^\circ C \overset{v_1}{\to} \cdots \overset{v_n}{\to} C^\circ$ is a full execution of C, then $M = \varphi({}^\circ C) \overset{\varphi(v_1)}{\longrightarrow} \cdots \overset{\varphi(v_n)}{\longrightarrow} \varphi(C^\circ) = \widetilde{M}$ is a sequence of transition firings in N.

Definition 2.10 *A* fireable in marking M process *of a net* N *is a pair* $\pi = (C, \varphi)$, *where* C *is a causal net and* $\varphi : C \to N$ *is an embedding s.t.* $M = \varphi({}^\circ C)$. *A fireable in* M_N process *is a* process *of* N.

We write $\Pi(N, M)$ for a *set of all fireable* in marking M processes of a net N and $\Pi(N)$ for the *set of all processes* of a net N. The *initial* process of a net N is $\pi_N = (C_N, \varphi_N) \in \Pi(N)$, s.t. $T_{C_N} = \emptyset$. If $\pi \in \Pi(N, M)$, then firing of this process transforms a marking M into $\widetilde{M} = M - \varphi(^\circ C) + \varphi(C^\circ) = \varphi(C^\circ)$, denoted by $M \xrightarrow{\pi} \widetilde{M}$.

Let $\pi = (C, \varphi)$, $\tilde{\pi} = (\widetilde{C}, \tilde{\varphi}) \in \Pi(N)$, $\hat{\pi} = (\widehat{C}, \hat{\varphi}) \in \Pi(N, \varphi(C^\circ))$. A process π is a *prefix* of a process $\tilde{\pi}$, if $T_C \subseteq T_{\widetilde{C}}$ is a left-closed set in $\widetilde{C}$. A process $\hat{\pi}$ is a *suffix* of a process $\tilde{\pi}$, if $T_{\widehat{C}} = T_{\widetilde{C}} \setminus T_C$. In such a case a process $\tilde{\pi}$ is an *extension* of π *by process* $\hat{\pi}$, and $\hat{\pi}$ is an *extending* process for π, denoted by $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$. We write $\pi \to \tilde{\pi}$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$ for some $\hat{\pi}$.

A process $\tilde{\pi}$ is an extension of a process π *by one transition*, denoted by $\pi \xrightarrow{v} \tilde{\pi}$ or $\pi \xrightarrow{a} \tilde{\pi}$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$, $T_{\widehat{C}} = \{v\}$ and $l_{\widehat{C}}(v) = a$.

A process $\tilde{\pi}$ is an extension of a process π *by sequence of transitions*, denoted by $\pi \xrightarrow{\sigma} \tilde{\pi}$ or $\pi \xrightarrow{\omega} \tilde{\pi}$, if $\exists \pi_i \in \Pi(N)$ $(1 \le i \le n)$ $\pi \xrightarrow{v_1} \pi_1 \xrightarrow{v_2} \dots \xrightarrow{v_n} \pi_n = \tilde{\pi}$, $\sigma = v_1 \cdots v_n$ and $l_{\widehat{C}}(\sigma) = \omega$.

A process $\tilde{\pi}$ is an extension of a process π *by multiset of transitions*, denoted by $\pi \xrightarrow{V} \tilde{\pi}$ or $\pi \xrightarrow{A} \tilde{\pi}$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$, $\prec_{\widehat{C}} = \emptyset$, $T_{\widehat{C}} = V$ and $l_{\widehat{C}}(V) = A$.

2.7 Branching processes

A branching process [11] is an extension of a notion of a (usual, deterministic) one s.t. there may exist alternative events in it. So, it may be considered as a "compilation" of different computations in the only unit, allowing one to observe all the interplays of events and take into account both causality and nondeterminism on equal basis.

Definition 2.11 *An* occurrence net *is an acyclic ordinary labeled net* $O = \langle P_O, T_O, F_O, l_O \rangle$, *s.t.:*

1. $\forall r \in P_O$ $|^\bullet r| \le 1$, *i.e. there are no backwards conflicts;*

2. $\forall x \in P_O \cup T_O$ $\neg(x \#_O x)$, *i.e. conflict relation is irreflexive;*

3. $\forall x \in P_O \cup T_O$ $|\downarrow_O x| < \infty$, *i.e. set of causes is finite.*

Let us note that on the basis of any occurrence net O one can define LES $\xi_O = \langle T_O, \prec_O \cap (T_O \times T_O), \#_O \cap (T_O \times T_O), l_O \rangle$.

Let $O = \langle P_O, T_O, F_O, l_O \rangle$ be occurrence net and $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$ be some net.

Definition 2.12 *A mapping* $\psi : P_O \cup T_O \to P_N \cup T_N$ *is an* embedding O into N, *notation* $\psi : O \to N$, *if:*

1. $\psi(P_O) \in \mathcal{M}(P_N)$ *and* $\psi(T_O) \in \mathcal{M}(T_N)$, *i.e. sorts are preserved;*

2. $\forall v \in T_O$ $l_O(v) = l_N(\psi(v))$, *i.e. labeling is preserved;*

3. $\forall v \in T_O$ $^\bullet \psi(v) = \psi(^\bullet v)$ *and* $\psi(v)^\bullet = \psi(v^\bullet)$, *i.e. flow relation is respected;*

4. $\forall v, w \in T_O$ $(^\bullet v = {}^\bullet w) \wedge (\psi(v) = \psi(w)) \Rightarrow v = w$, *i.e. there are no "superfluous" conflicts.*

Definition 2.13 *A branching process of a net* N *is a pair* $\varpi = (O, \psi)$, *where* O *is an occurrence net and* $\psi : O \to N$ *is an embedding s.t.* $M_N = \psi(^\circ O)$.

We write $\wp(N)$ for a set of *all branching processes* of a net N. The *initial* branching process of a net N coincides with it's initial process, i.e. $\varpi_N = \pi_N$.

Let $\varpi = (O, \psi)$, $\tilde{\varpi} = (\widetilde{O}, \tilde{\psi}) \in \wp(N)$, $O = \langle P_O, T_O, F_O, l_O \rangle$, $\widetilde{O} = \langle P_{\widetilde{O}}, T_{\widetilde{O}}, F_{\widetilde{O}}, l_{\widetilde{O}} \rangle$. A branching process ϖ is a *prefix* of a process $\tilde{\varpi}$, if $T_O \subseteq T_{\widetilde{O}}$ is a left-closed set in $\widetilde{O}$. In such a case branching process $\tilde{\varpi}$ is an *extension* of ϖ, and $\hat{\varpi}$ is an *extending* branching process for ϖ, denoted by $\varpi \to \tilde{\varpi}$.

A branching process ϖ of a net N is *maximal*, if it cannot be extended, i.e. $\forall \varpi = (O, \psi)$ s.t. $\varpi \to \tilde{\varpi}$: $T_{\widetilde{O}} \setminus T_O = \emptyset$. A set of all maximal branching processes of a net N consists of the unique (up to isomorphism) branching process $\varpi_{max} = (O_{max}, \psi_{max})$. In such a case an isomorphism class of occurrence net O_{max} is an *unfolding* of a net N, notation $\mathcal{U}(N)$. On the basis of unfolding $\mathcal{U}(N)$ of a net N one can define MES $\mathcal{E}(N) = \xi_{\mathcal{U}(N)}$ which is an isomorphism class of LES ξ_O for $O \in \mathcal{U}(N)$.

3 Basic τ-equivalences

In this section, we propose basic τ-equivalences: trace, bisimulation and conflict preserving. They will form a basic "frame" of relations for our further investigation.

3.1 τ-trace equivalences

Trace equivalences are the simplest ones. In trace semantics, a behavior of a system is associated with the set of all possible sequences of activities, i.e. protocols of work or computations. Thus, the points of nondeterministic choice between several extensions of a particular computation are not taken into account.

Let us introduce formal definitions of the trace relations.

We denote the empty string by the symbol ε.

Let $\sigma = a_1 \cdots a_n \in Act_\tau^*$. We define $vis(\sigma)$ as follows (in the following definition $a \in Act_\tau$).

1. $vis(\varepsilon) = \varepsilon$;

2. $vis(\sigma a) = \begin{cases} vis(\sigma)a, & a \neq \tau; \\ vis(\sigma), & a = \tau. \end{cases}$

Definition 3.1 A visible interleaving trace *of a net N is a sequence* $vis(a_1 \cdots a_n) \in Act^*$ *s.t.* $\pi_N \xrightarrow{a_1} \pi_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} \pi_n$, $\pi_i \in \Pi(N)$ $(1 \leq i \leq n)$. *We denote a set of* all visible interleaving traces *of a net N by* $VisIntTraces(N)$. *Two nets N and N' are* interleaving τ-trace equivalent, *denoted by* $N \equiv_i^\tau N'$, *if* $VisIntTraces(N) = VisIntTraces(N')$.

Let $\Sigma = A_1 \cdots A_n \in (\mathcal{M}(Act_\tau))^*$. We define $vis(\Sigma)$ as follows (in the following definition $A \in \mathcal{M}(Act_\tau)$).

1. $vis(\varepsilon) = \varepsilon$;

2. $vis(\Sigma A) = \begin{cases} vis(\Sigma)(A \cap Act), & A \cap Act \neq \emptyset; \\ vis(\Sigma), & \text{otherwise.} \end{cases}$

Definition 3.2 A visible step trace *of a net N is a sequence* $vis(A_1 \cdots A_n) \in (\mathcal{M}(Act))^*$ *s.t.* $\pi_N \xrightarrow{A_1} \pi_1 \xrightarrow{A_2} \ldots \xrightarrow{A_n} \pi_n$, $\pi_i \in \Pi(N)$ $(1 \leq i \leq n)$. *We denote a set of* all visible step traces *of a net N by* $VisStepTraces(N)$. *Two nets N and N' are* step τ-trace equivalent, *denoted by* $N \equiv_s^\tau N'$, *if* $VisStepTraces(N) = VisStepTraces(N')$.

Let $\rho = \langle X, \prec, l \rangle$ is lposet s.t. $l : X \to Act_\tau$. We denote $vis(X) = \{x \in X \mid l(x) \in Act\}$ and $vis(\rho) = \rho|_{vis(X)}$.

Definition 3.3 A visible pomset trace *of a net N is a pomset* $vis(\rho)$, *an isomorphism class of lposet* $vis(\rho_C)$ *for* $\pi = (C, \varphi) \in \Pi(N)$. *We denote a set of* all visible pomset traces *of a net N by* $VisPomsets(N)$. *Two nets N and N' are* partial word τ-trace equivalent, *denoted by* $N \equiv_{pw}^\tau N'$, *if* $VisPomsets(N) \sqsubseteq VisPomsets(N')$ *and* $VisPomsets(N') \sqsubseteq VisPomsets(N)$.

Definition 3.4 *Two nets N and N' are* pomset τ-trace equivalent, *denoted by* $N \equiv_{pom}^\tau N'$, *if* $VisPomsets(N) = VisPomsets(N')$.

3.2 τ-bisimulation equivalences

Bisimulation equivalences completely respect points of nondeterministic choice in the behavior of a modeled system, unlike trace ones.

Let $C = \langle P_C, T_C, F_C, l_C \rangle$ be causal net. We denote $vis(T_C) = \{v \in T_C \mid l_C(v) \in Act\}$ and $vis(\prec_C) = \prec_C \cap (vis(T_C) \times vis(T_C))$.

3.2.1 Usual τ-bisimulation equivalences

Usual bisimulation equivalences are the simplest (and weakest) ones in the bisimulation semantics. They require a mutual simulation of the parts of a "new" computations which extend the "present" ones, i.e. "extending" parts.

Definition 3.5 *Let N and N' be some nets. A relation* $\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$ *is a* $\star$-τ-bisimulation *between N and N', $\star \in$ {interleaving, step, partial word, pomset}, denoted by* $\mathcal{R} : N \underleftrightarrow{}_\star^\tau N'$, $\star \in \{i, s, pw, pom\}$, *if:*

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$.

2. $(\pi, \pi') \in \mathcal{R}$, $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$,

    (a) $|vis(T_{\widehat{C}})| = 1$, *if* $\star = i$;
    (b) $vis(\prec_{\widehat{C}}) = \emptyset$, *if* $\star = s$;

$\Rightarrow \exists \tilde{\pi}': \ \pi' \xrightarrow{\hat{\tilde{\pi}}'} \tilde{\pi}', \ (\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$ and

(a) $vis(\rho_{\widehat{C'}}) \sqsubseteq vis(\rho_{\widehat{C}})$, if $\star = pw$;
(b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C'}})$, if $\star \in \{i, s, pom\}$.

3. As item 2, but the roles of N and N' are reversed.

Two nets N and N' are $\star$-τ-bisimulation equivalent, $\star \in$ {interleaving, step, partial word, pomset}, denoted by $N\underleftrightarrow{\tau}_{\star}N'$, if $\exists \mathcal{R}: N\underleftrightarrow{\tau}_{\star}N'$, $\star \in \{i, s, pw, pom\}$.

3.2.2 ST-τ-bisimulation equivalences

ST-bisimulation equivalences respect (in some sense) the duration of event occurrences in a computation supposing that these events happen not instantaneously, but have the beginning and the end. The relations require a mutual simulation of extending parts of computations *plus* the parts consisting of events which are active now (i.e. that which have begun but have not finished yet).

We begin with the definition of ST-process which is a special structure containing the information about as causal dependencies of events in the present computation as the events which finished their work and are not active at the present moment.

Definition 3.6 ST-τ-process *of a net N is a pair* (π_E, π_P) *s.t.* $\pi_E, \pi_P \in \Pi(N)$, $\pi_P \xrightarrow{\pi_W} \pi_E$ *and* $\forall v, w \in T_{C_E}$ $(v \prec_{C_E} w) \vee (l_{C_E}(v) = \tau) \Rightarrow v \in T_{C_P}$.

In such a case π_E is a process which began working, π_P corresponds to the completed part of π_E, and π_W — to the still working part. Obviously, $\prec_{C_W} = \emptyset$. We denote a set of *all ST-τ-processes* of a net N by $ST^{\tau} - \Pi(N)$. (π_N, π_N) is the *initial ST-τ-process* of a net N. Let $(\pi_E, \pi_P), (\tilde{\pi}_E, \tilde{\pi}_P) \in ST^{\tau} - \Pi(N)$. We write $(\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P)$, if $\pi_E \to \tilde{\pi}_E$ and $\pi_P \to \tilde{\pi}_P$.

Definition 3.7 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^{\tau} - \Pi(N) \times ST^{\tau} - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}), \ \pi = (C, \varphi) \in \Pi(N), \ \pi' = (C', \varphi') \in \Pi(N')\}$ is a $\star$-ST-τ-bisimulation between N and N', $\star \in$ {interleaving, partial word, pomset}, denoted by $\mathcal{R}: N\underleftrightarrow{\tau}_{\star ST}N'$, $\star \in \{i, pw, pom\}$, if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.

2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \Rightarrow \beta : vis(\rho_{C_E}) \asymp vis(\rho_{C'_E})$ and $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.

3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}, \ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \Rightarrow \exists \tilde{\beta}, \ (\tilde{\pi}'_E, \tilde{\pi}'_P): \ (\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P), \ \tilde{\beta}|_{vis(T_{C_E})} = \beta, \ ((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$, and if $\pi_P \xrightarrow{\pi} \tilde{\pi}_E, \ \pi'_P \xrightarrow{\pi'} \tilde{\pi}'_E, \ \gamma = \tilde{\beta}|_{vis(T_C)}$, then:

   (a) $\gamma^{-1} : vis(\rho_{C'}) \sqsubseteq vis(\rho_C)$, if $\star = pw$;
   (b) $\gamma : vis(\rho_C) \simeq vis(\rho_{C'})$, if $\star = pom$.

4. As item 3, but the roles of N and N' are reversed.

Two nets N and N' are $\star$-ST-τ-bisimulation equivalent, $\star \in$ {interleaving, partial word, pomset}, denoted by $N\underleftrightarrow{\tau}_{\star ST}N'$, if $\exists \mathcal{R}: N\underleftrightarrow{\tau}_{\star ST}N'$, $\star \in \{i, pw, pom\}$.

3.2.3 History preserving τ-bisimulation equivalences

History preserving bisimulation equivalences respect "histories" of work, i.e. require a mutual modeling of the *whole* computations, from the beginning to the end.

Definition 3.8 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}), \ \pi = (C, \varphi) \in \Pi(N), \ \pi' = (C', \varphi') \in \Pi(N')\}$, is a pomset history preserving τ-bisimulation between N and N', denoted by $\mathcal{R}: N\underleftrightarrow{\tau}_{pomh}N'$, if:*

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.

2. $(\pi, \pi', \beta) \in \mathcal{R} \Rightarrow \beta : vis(\rho_C) \simeq vis(\rho_{C'})$.

3. $(\pi, \pi', \beta) \in \mathcal{R}, \ \pi \to \tilde{\pi} \Rightarrow \exists \tilde{\beta}, \ \tilde{\pi}' : \pi' \to \tilde{\pi}', \ \tilde{\beta}|_{vis(T_C)} = \beta, \ (\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{R}$.

4. As item 3, but the roles of N and N' are reversed.

Two nets N and N' are pomset history preserving τ-bisimulation equivalent, denoted by $N\underleftrightarrow{\tau}_{pomh}N'$, if $\exists \mathcal{R}: N\underleftrightarrow{\tau}_{pomh}N'$.

3.2.4 History preserving ST-τ-bisimulation equivalences

History preserving ST-bisimulation equivalences may be considered as modification of history preserving ones s.t. the beginnings and the ends of events are taken into account.

Definition 3.9 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$, is a* pomset history preserving ST-τ-bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhST} N'$, if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.

2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \Rightarrow \beta : vis(\rho_{C_E}) \simeq vis(\rho_{C'_E})$ *and* $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.

3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \Rightarrow \exists \tilde{\beta},\ (\tilde{\pi}'_E, \tilde{\pi}'_P) : (\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),\ \tilde{\beta}|_{vis(T_{C_E})} = \beta,\ ((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$.

4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are pomset history preserving ST-τ-bisimulation equivalent, *denoted by $N \underline{\leftrightarrow}^\tau_{pomhST} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhST} N'$.*

3.2.5 Usual branching τ-bisimulation equivalences

Usual branching bisimulation equivalences are the simplest of branching bisimulation ones and may be considered as a modification of a notion of usual bisimulation. The word "branching" is used to indicate that these relations "really" respect all aspects of branching with special care for silent actions. Note that (non-branching) bisimulation notions take no special care for silent actions in the points of nondeterministic choice, but such actions may play an important role in the behavior of a modeled system.

In Figure 1, a distinguish ability of the usual and the branching bisimulation equivalences is demonstrated for two nets N and N'. All these equivalences require the initial processes π_N and $\pi_{N'}$ to be related by bisimulation. Further, if present processes π and π' are bisimilar, and one of them is extended, then the process of another net can be extended so that to model the behavior of the first net abstracting from invisible actions. In such a case, the new, extended processes $\tilde{\pi}$ and $\tilde{\pi}$ should be also bisimilar.

Branching τ-bisimulation equivalences are more strict than usual ones, since they require that some intermediate processes should be also bisimilar. An extension by invisible action τ, represented in Figure 1(a), is simulated by an extension by sequence of invisible actions. In addition, the new process $\tilde{\pi}$ of the first net should be related with the present process π of the second net. An extension by visible action a, depicted in Figure 1(b), is simulated by an extension by a sequence of actions s.t. only one of them (namely, a) is visible. In addition, the present process π should be related with π_1 which is reached immediately before the extension by an action a. The new process $\tilde{\pi}$ should be bisimilar with π_2 which is reached immediately after the extension by an action a. These additional relations characterizing a notion of "branching" are depicted by dashed lines in Figure 1.

For some net N and $\pi, \tilde{\pi} \in \Pi(N)$ we write $\pi \Rightarrow \tilde{\pi}$ when $\exists \hat{\pi} = (\widehat{C}, \hat{\varphi})$ s.t. $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$ and $vis(T_{\widehat{C}}) = \emptyset$.

Definition 3.10 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$ is an* interleaving branching τ-bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{ibr} N'$, if:*

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$.

2. $(\pi, \pi') \in \mathcal{R},\ \pi \xrightarrow{a} \tilde{\pi} \Rightarrow$

    (a) $a = \tau$ *and* $(\tilde{\pi}, \pi') \in \mathcal{R}$ *or*

    (b) $a \neq \tau$ *and* $\exists \bar{\pi}', \tilde{\pi}' : \pi' \Rightarrow \bar{\pi}' \xrightarrow{a} \tilde{\pi}',\ (\pi, \bar{\pi}') \in \mathcal{R},\ (\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$.

3. *As item 2, but the roles of N and N' are reversed.*

Two nets N and N' are interleaving branching τ-bisimulation equivalent, *denoted by $N \underline{\leftrightarrow}^\tau_{ibr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{ibr} N'$.*
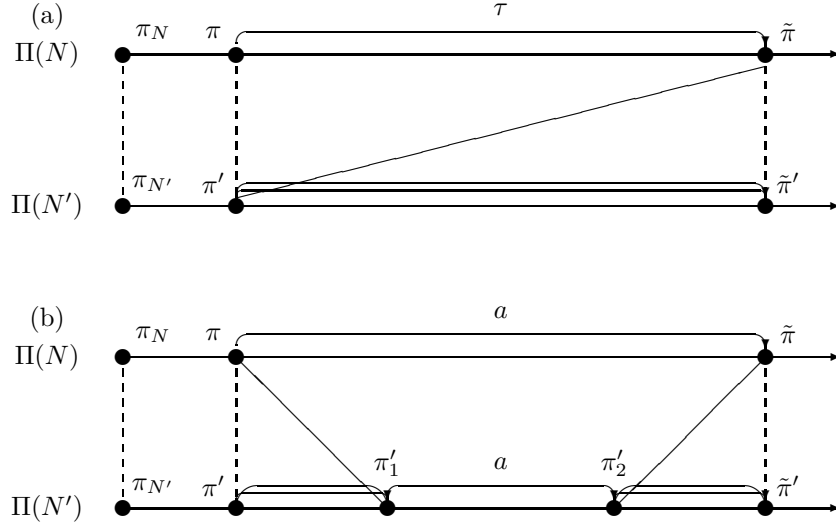
Figure 1: A distinguish ability of the usual and the branching τ-bisimulation equivalences

3.2.6 History preserving branching τ-bisimulation equivalences

History preserving branching bisimulation equivalences are modifications of history preserving bisimulation ones in accordance to the special "branching" idea.

Definition 3.11 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : T_C \to T_{C'}, \ \pi = (C, \varphi) \in \Pi(N), \ \pi' = (C', \varphi') \in \Pi(N')\}$, is a pomset history preserving branching τ-bisimulation between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$, if:*

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.

2. $(\pi, \pi', \beta) \in \mathcal{R} \ \Rightarrow \ beta : vis(\rho_C) \simeq vis(\rho_{C'})$.

3. $(\pi, \pi', \beta) \in \mathcal{R}, \ \pi \to \tilde{\pi} \ \Rightarrow$

   (a) $(\tilde{\pi}, \pi', \beta) \in \mathcal{R}$ or
   (b) $\exists \tilde{\beta}, \ \bar{\pi}', \ \tilde{\pi}' : \pi' \Rightarrow \bar{\pi}' \to \tilde{\pi}', \ \tilde{\beta}|_{vis(T_C)} = \beta, \ (\pi, \bar{\pi}', \beta) \in \mathcal{R}, \ (\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{R}$.

4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are pomset history preserving branching τ-bisimulation equivalent, *denoted by $N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$.*

3.2.7 ST-branching τ-bisimulation equivalences

ST-branching bisimulation equivalences are modifications of ST-bisimulation ones in accordance to the "branching" idea.

Let $(\pi_E, \pi_P), \ (\tilde{\pi}_E, \tilde{\pi}_P) \in ST^{\tau} - \Pi(N)$. We write $(\pi_E, \pi_P) \Rightarrow (\tilde{\pi}_E, \tilde{\pi}_P)$, if $\pi_E \Rightarrow \tilde{\pi}_E$ and $\pi_P \Rightarrow \tilde{\pi}_P$.

Definition 3.12 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^{\tau} - \Pi(N) \times ST^{\tau} - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}), \ \pi = (C, \varphi) \in \Pi(N), \ \pi' = (C', \varphi') \in \Pi(N')\}$ is an interleaving ST-branching τ-bisimulation between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{iSTbr} N'$, if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.

2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \ \Rightarrow \ \beta : vis(\rho_{C_E}) \asymp vis(\rho_{C'_E})$ and $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.

3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}, \ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \ \Rightarrow$

   (a) $((\tilde{\pi}_E, \tilde{\pi}_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$ or

(b) $\exists \tilde{\beta}, \ (\bar{\pi}'_E, \bar{\pi}'_P), \ (\tilde{\pi}'_E, \tilde{\pi}'_P) : \ (\pi'_E, \pi'_P) \Rightarrow (\bar{\pi}'_E, \bar{\pi}'_P) \rightarrow (\tilde{\pi}'_E, \tilde{\pi}'_P), \ \tilde{\beta}|_{vis(T_{C_E})} = \beta, \ ((\pi_E, \pi_P), (\bar{\pi}'_E, \bar{\pi}'_P), \beta) \in \mathcal{R}, \ ((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}.$

4. As item 3, but the roles of N and N' are reversed.

Two nets N and N' are interleaving ST-branching τ-bisimulation equivalent, denoted by $N \underline{\leftrightarrow}^\tau_{iSTbr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{iSTbr} N'$.

3.2.8 History preserving ST-branching τ-bisimulation equivalences

History preserving ST-branching bisimulation equivalences are modifications of history preserving ST-bisimulation ones in accordance to the "branching" idea.

Definition 3.13 Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \rightarrow vis(T_{C'}), \ \pi = (C, \varphi) \in \Pi(N), \ \pi' = (C', \varphi') \in \Pi(N')\}$ is a pomset history preserving ST-branching τ-bisimulation between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$, if:

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}.$

2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \Rightarrow \beta : vis(\rho_{C_E}) \simeq vis(\rho_{C'_E})$ and $\beta(vis(T_{C_P})) = vis(T_{C'_P}).$

3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}, \ (\pi_E, \pi_P) \rightarrow (\tilde{\pi}_E, \tilde{\pi}_P) \Rightarrow$

    (a) $((\tilde{\pi}_E, \tilde{\pi}_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$ or

    (b) $\exists \tilde{\beta}, \ (\bar{\pi}'_E, \bar{\pi}'_P), \ (\tilde{\pi}'_E, \tilde{\pi}'_P) : \ (\pi'_E, \pi'_P) \Rightarrow (\bar{\pi}'_E, \bar{\pi}'_P) \rightarrow (\tilde{\pi}'_E, \tilde{\pi}'_P), \ \tilde{\beta}|_{vis(T_{C_E})} = \beta, \ ((\pi_E, \pi_P), (\bar{\pi}'_E, \bar{\pi}'_P), \beta) \in \mathcal{R}, \ ((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}.$

4. As item 3, but the roles of N and N' are reversed.

Two nets N and N' are pomset history preserving ST-branching τ-bisimulation equivalent, denoted by $N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$.

3.3 Conflict preserving τ-equivalences

Conflict preserving equivalences copmletely respect conflicts in the behavior of a modeled system. The behavior is associated with the event structure.

Let $\xi = \langle X, \prec, \#, l \rangle$ be a LES s.t. $l : X \rightarrow Act_\tau$. We denote $vis(X) = \{x \in X \mid l(x) \in Act\}$ and $vis(\xi) = \xi|_{vis(X)}.$

Definition 3.14 Two nets N and N' are MES conflict preserving τ-equivalent, denoted by $N \equiv^\tau_{mes} N'$, if $vis(\mathcal{E}(N)) = vis(\mathcal{E}(N')).$

3.4 Interrelations of basic τ-equivalences

In this section, we compare basic τ-equivalences and obtain the lattice of their interrelations as a result.

In the following, the symbol '_' will denote "nothing", and the signs of equivalences subscribed by it are considered as that of without any subscribtion.

Theorem 3.1 Let $\leftrightarrow, \leftrightarrow\hspace{-1.2mm}\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau, \simeq\}$, $\star, \star\star \in \{\_, i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr, pomhbr, iSTbr, pomhSTbr, mes\}$. For nets N and N' $N \leftrightarrow_\star N' \Rightarrow N \leftrightarrow\hspace{-1.2mm}\leftrightarrow_{\star\star} N'$ iff in the graph in Figure 2 there exists a directed path from $\leftrightarrow_\star$ to $\leftrightarrow\hspace{-1.2mm}\leftrightarrow_{\star\star}$.

Proof. ($\Leftarrow$) Let us check the validity of the implications in the graph in Figure 2.

- The implications $\leftrightarrow^\tau_s \rightarrow \leftrightarrow^\tau_i$, $\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau\}$, are valid since isomorphism of lposets with empty precedence relation is isomorphism of singleton ones.

- The implications $\leftrightarrow^\tau_{pw} \rightarrow \leftrightarrow^\tau_s$, $\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau\}$, are valid since homomorphism of lposets is isomorphism of lposets with empty precedence relation.

- The implication $\underline{\leftrightarrow}^\tau_{pwST} \rightarrow \underline{\leftrightarrow}^\tau_{iST}$ is valid since homomorphism of lposets is isomorphism of singleton ones.

- The implications $\leftrightarrow^\tau_{pom} \rightarrow \leftrightarrow^\tau_{pw}$, $\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau\}$, are valid since isomorphism of lposets is homomorphism.
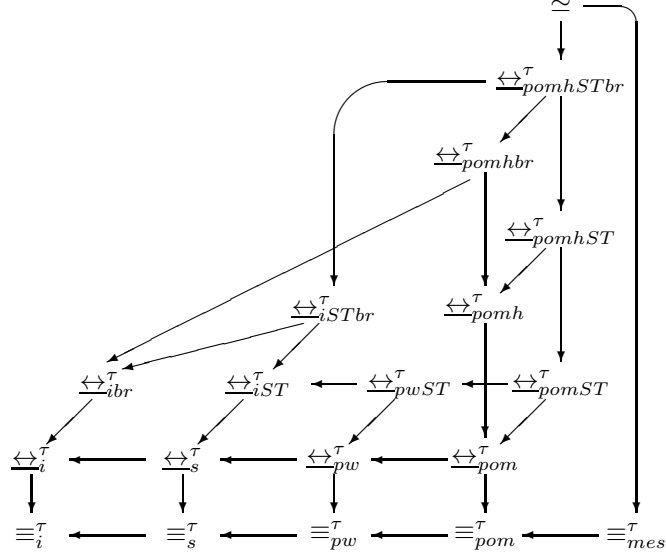
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Figure 2: Interrelations of basic τ-equivalences

- The implication $\equiv_{mes}^{\tau} \rightarrow \equiv_{pom}^{\tau}$ is valid since isomorphic LES's have isomorphic sets of lposets.

- The implication $\underleftrightarrow{}_{i}^{\tau} \rightarrow \equiv_{i}^{\tau}$ is proved as follows. Let $\mathcal{R}: N\underleftrightarrow{}_{i}^{\tau}N'$. If $\pi_N \overset{a_1}{\rightarrow} \pi_1 \overset{a_2}{\rightarrow} \ldots \overset{a_n}{\rightarrow} \pi_n$, then there exists a sequence $(\pi_N, \pi_{N'}), \ldots, (\pi_n, \pi'_m) \in \mathcal{R}$ s.t. $\pi_{N'} \overset{a'_1}{\rightarrow} \pi'_1 \overset{a'_2}{\rightarrow} \ldots \overset{a'_m}{\rightarrow} \pi'_m$, $vis(a_1 \cdots a_n) = vis(a'_1 \cdots a'_m)$, and vice versa, due to the symmetry of bisimulation.

- The implication $\underleftrightarrow{}_{s}^{\tau} \rightarrow \equiv_{s}^{\tau}$ is proved as the previous one but with use of $A_1, \ldots, A_n \in \mathcal{M}(Act_\tau)$ instead of $a_1, \ldots, a_n \in Act_\tau$.

- The implication $\underleftrightarrow{}_{pw}^{\tau} \rightarrow \equiv_{pw}^{\tau}$ is proved as follows. Let $\mathcal{R}: N\underleftrightarrow{}_{pw}^{\tau}N'$ and $\pi = (C, \varphi) \in \Pi(N)$. Since $\pi_N \overset{\pi}{\rightarrow} \pi$, then $\exists (\pi, \pi') \in \mathcal{R}$ s.t. $\pi' = (C', \varphi')$ and $vis(\rho_{C'}) \sqsubseteq vis(\rho_C)$. Hence, $VisPomsets(N') \sqsubseteq VisPomsets(N)$. The inclusion $VisPomsets(N) \sqsubseteq VisPomsets(N')$ is proved similarly, due to the symmetry of bisimulation.

- The implication $\underleftrightarrow{}_{pom}^{\tau} \rightarrow \equiv_{pom}^{\tau}$ is proved as the previous one but with use of isomorphism instead of homomorphism.

- The implication $\underleftrightarrow{}_{iST}^{\tau} \rightarrow \underleftrightarrow{}_{s}^{\tau}$ is proved as previous ones with use of the fact that a step $\pi \overset{A}{\rightarrow} \tilde{\pi}$, where $A = \{a_1, \ldots, a_n\} \in \mathcal{M}(Act)$, corresponds to the sequence of ST-τ-processes $(\pi_0, \pi_0), \ldots, (\pi_n, \pi_0), \ldots, (\pi_n, \pi_n)$ s.t. $\pi = \pi_0 \overset{a_1}{\rightarrow} \ldots \overset{a_n}{\rightarrow} \pi_n = \tilde{\pi}$.

- The implications $\underleftrightarrow{}_{\star ST}^{\tau} \rightarrow \underleftrightarrow{}_{\star}^{\tau}$, $\star \in \{pw, pom\}$ are proved with constructing on the basis of the relation $\mathcal{R}: N\underleftrightarrow{}_{\star ST}^{\tau}N'$ the new relation $\mathcal{S}: N\underleftrightarrow{}_{\star}^{\tau}N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \beta\ ((\pi, \pi), (\pi', \pi'), \beta) \in \mathcal{R}\}$.

- The implication $\underleftrightarrow{}_{pomhST}^{\tau} \rightarrow \underleftrightarrow{}_{pomh}^{\tau}$ is proved with constructing on the basis of the relation $\mathcal{R}: N\underleftrightarrow{}_{pomhST}^{\tau}N'$ the new relation $\mathcal{S}: N\underleftrightarrow{}_{pomh}^{\tau}N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi', \beta) \mid ((\pi, \pi), (\pi', \pi'), \beta) \in \mathcal{R}\}$.

- The implication $\underleftrightarrow{}_{pomh}^{\tau} \rightarrow \underleftrightarrow{}_{pom}^{\tau}$ is proved with constructing on the basis of the relation $\mathcal{R}: N\underleftrightarrow{}_{pomh}^{\tau}N'$ the new relation $\mathcal{S}: N\underleftrightarrow{}_{pom}^{\tau}N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \beta\ ((\pi, \pi), (\pi', \pi'), \beta) \in \mathcal{R}\}$.

- The implication $\underleftrightarrow{}_{pomhST}^{\tau} \rightarrow \underleftrightarrow{}_{pomST}^{\tau}$ follows from the definitions.

- The implication $\underleftrightarrow{}_{ibr}^{\tau} \rightarrow \underleftrightarrow{}_{i}^{\tau}$ follows from the definitions.

- The implication $\underleftrightarrow{}_{pomhbr}^{\tau} \rightarrow \underleftrightarrow{}_{pomh}^{\tau}$ follows from the definitions.

- The implication $\underleftrightarrow{}_{pomhbr}^{\tau} \rightarrow \underleftrightarrow{}_{ibr}^{\tau}$ is proved with constructing on the basis of the relation $\mathcal{R}: N\underleftrightarrow{}_{pomhbr}^{\tau}N'$ the new relation $\mathcal{S}: N\underleftrightarrow{}_{ibr}^{\tau}N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \beta\ (\pi, \pi', \beta) \in \mathcal{R}\}$.

- The implication $\underline{\leftrightarrow}^\tau_{iSTbr} \rightarrow \underline{\leftrightarrow}^\tau_{ibr}$ is proved with constructing on the basis of the relation $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{iSTbr} N'$ the new relation $\mathcal{S} : N \underline{\leftrightarrow}^\tau_{ibr} N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \beta \ (\pi, \pi', \beta) \in \mathcal{R}\}$.

- The implication $\underline{\leftrightarrow}^\tau_{iSTbr} \rightarrow \underline{\leftrightarrow}^\tau_{iST}$ follows from the definitions.

- The implication $\underline{\leftrightarrow}^\tau_{pomhSTbr} \rightarrow \underline{\leftrightarrow}^\tau_{iSTbr}$ follows from the definitions.

- The implication $\underline{\leftrightarrow}^\tau_{pomhSTbr} \rightarrow \underline{\leftrightarrow}^\tau_{pomhbr}$ is proved with constructing on the basis of the relation $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$ the new relation $\mathcal{S} : N \underline{\leftrightarrow}^\tau_{pomhbr} N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi', \beta) \mid ((\pi, \pi), (\pi', \pi'), \beta) \in \mathcal{R}\}$.

- The implication $\underline{\leftrightarrow}^\tau_{pomhSTbr} \rightarrow \underline{\leftrightarrow}^\tau_{pomhST}$ follows from the definitions.

- The implication $\simeq \rightarrow \underline{\leftrightarrow}^\tau_{pomhSTbr}$ is obvious.

- The implication $\simeq \rightarrow \equiv^\tau_{mes}$ is obvious.

($\Rightarrow$) An absence of additional nontrivial arrows in the graph in Figure 2 is proved by the following examples.

- In Figure 3(a), $N \underline{\leftrightarrow}^\tau_{ibr} N'$, but $N \not\equiv^\tau_s N'$, since only in the net N' actions a and b cannot happen concurrently.

- In Figure 3(c), $N \underline{\leftrightarrow}^\tau_{iSTbr} N'$, but $N \not\equiv^\tau_{pw} N'$, since for the pomset corresponding to the net N there is no even less sequential pomset in N'.

- In Figure 3(b), $N \underline{\leftrightarrow}^\tau_{pwST} N'$, but $N \not\equiv^\tau_{pom} N'$, since only in the net N' action b can depend on action a.

- In Figure 5(a), $N \equiv^\tau_{mes} N'$, but $N \not\underline{\leftrightarrow}^\tau_i N'$, since only in the net N' action τ can happen so that in the corresponding initial state of the net N action a cannot happen.

- In Figure 4(a), $N \underline{\leftrightarrow}^\tau_{pom} N'$, but $N \not\underline{\leftrightarrow}^\tau_{iST} N'$, since only in the net N' action a can start so that no action b can begin to work until finishing a.

- In Figure 4(b), $N \underline{\leftrightarrow}^\tau_{pomST} N'$, but $N \not\underline{\leftrightarrow}^\tau_{pomh} N'$, since only in the net N' after action a action b can happen so that action c must depend on a.

- In Figure 5(b), $N \underline{\leftrightarrow}^\tau_{pomh} N'$, but $N \not\underline{\leftrightarrow}^\tau_{iST} N'$, since only in the net N' action a can start so that the action b can never occur.

- In Figure 5(c), $N \underline{\leftrightarrow}^\tau_{pomhST} N'$, but $N \not\underline{\leftrightarrow}^\tau_{ibr} N'$, since in the net N' an action a can happen so that it will be simulated by sequence of actions τa in N. Then the state of the net N reached after τ must be related with the initial state of a net N, but in such a case the occurrence of action b from the initial state of N' cannot be imitated from the corresponding state of N.

- In Figure 5(d), $N \underline{\leftrightarrow}^\tau_{pomhbr} N'$, but $N \not\underline{\leftrightarrow}^\tau_{iST} N'$, since in the net N' an action c may start so that during work of the corresponding action c in the net N an action a may happen in such a way that the action b never occur.

- In Figure 4(c), $N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$, but $N \not\equiv^\tau_{mes} N'$, since only the MES corresponding to the net N' has two conflict actions a.

- In Figure 4(d), $N \equiv^\tau_{mes} N'$, but $N \not\simeq N'$, since unfireable transitions of the nets N and N' are labeled by different actions (a and b). $\square$

Thus, we obtained a number of interesting results.

In Petri nets with silent transitions ST- and history preserving equivalences are independent unlike the situation with their analogues on nets without silent transitions. Moreover, we have a new dimension of branching equivalences. So, we proposed additional notions of $\underline{\leftrightarrow}^\tau_{pomhST}$ and $\underline{\leftrightarrow}^\tau_{ibr}$, $\underline{\leftrightarrow}^\tau_{pomhbr}$.

In this paper, we obtained also two new notions $\underline{\leftrightarrow}^\tau_{iSTbr}$ and $\underline{\leftrightarrow}^\tau_{pomhSTbr}$ which are results of application of ST- and branching idea to both interleaving and pomset semantics.

In addition, the equivalence $\equiv^\tau_{mes}$ imply only trace equivalences, and no more, unlike on nets without silent transitions, where its analogue was the strongest notion in pomset semantics.
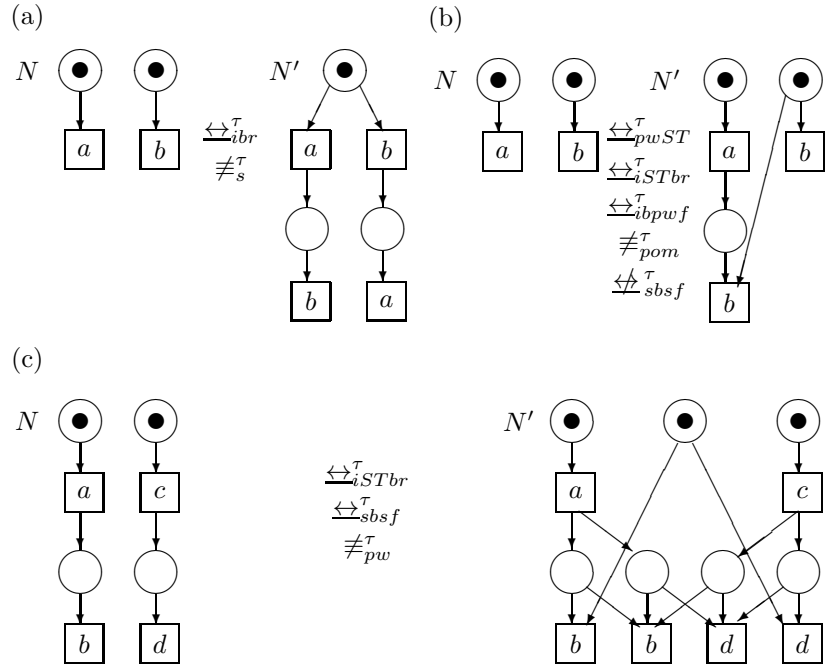
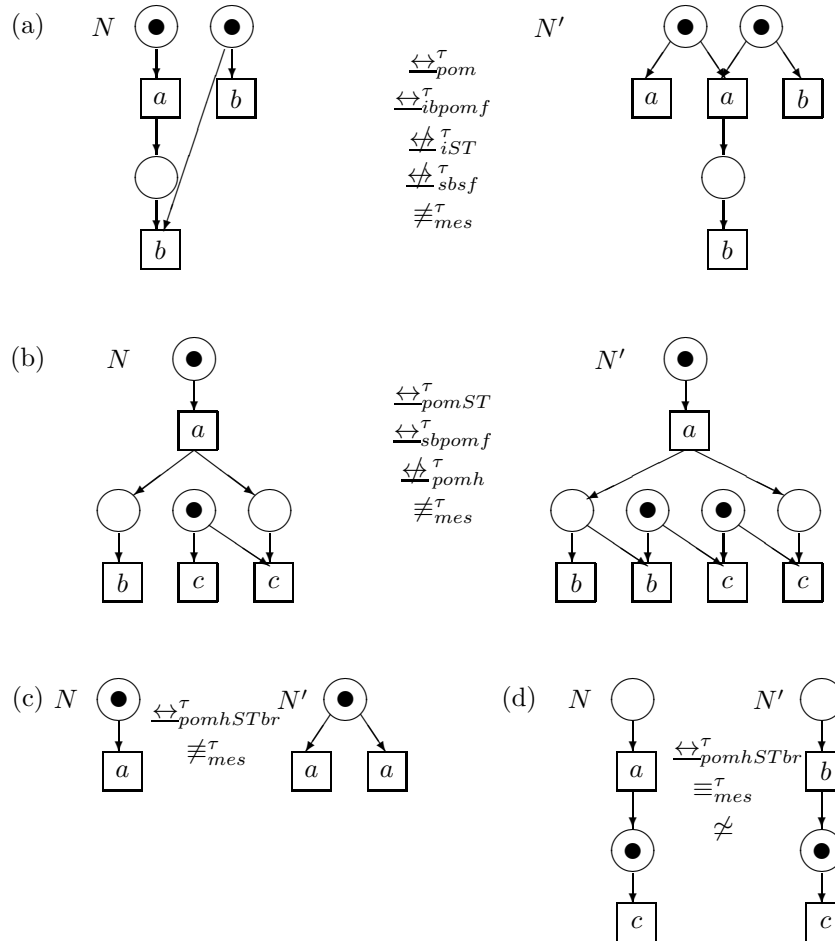Figure 3: Examples of basic τ-equivalences
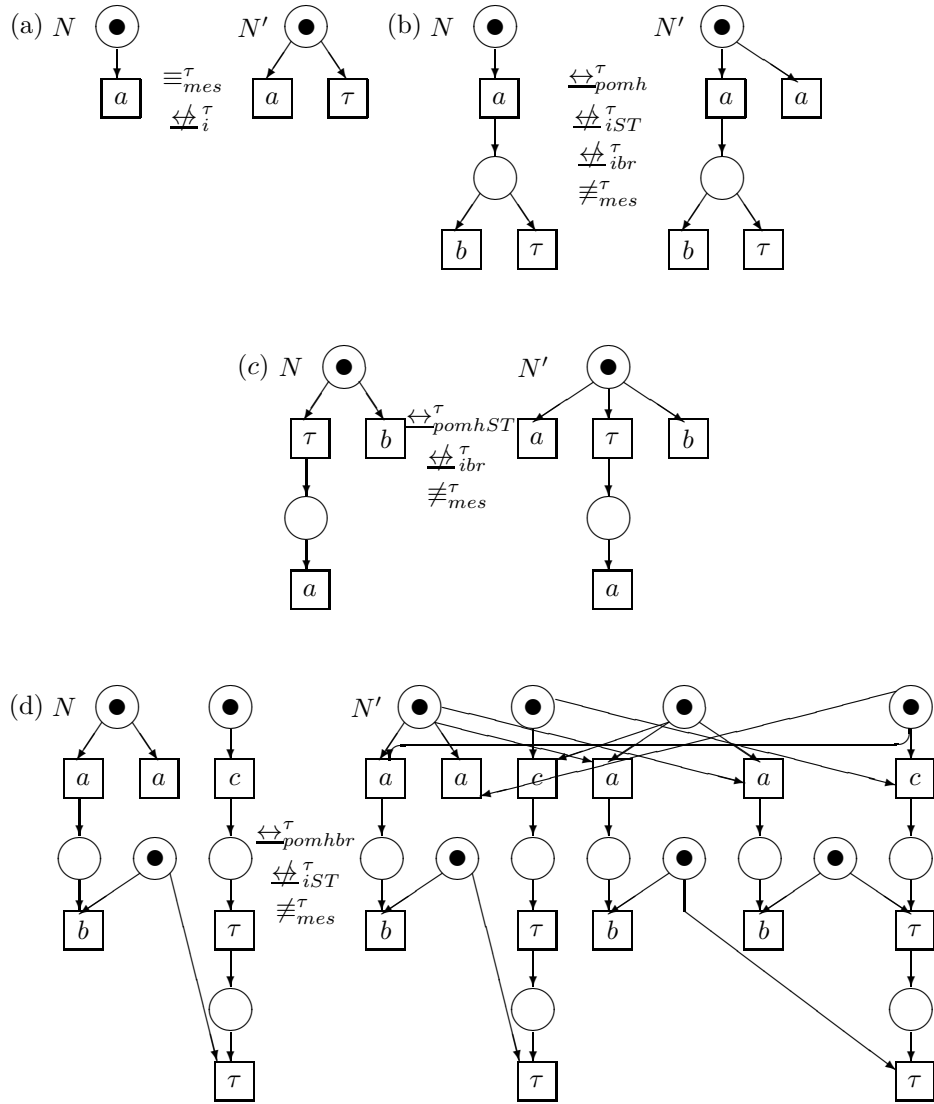

Figure 4: Examples of basic τ-equivalences (continued)

Figure 5: Examples of basic τ-equivalences (continued 2)

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4 Back-forth τ-bisimulation equivalences

In this section, we propose back-forth τ-bisimulation equivalences. The distinctive feature of these relations is that they require a mutual simulation not only in forward direction (as usual) but also in backward direction.

4.1 Sequential runs

A sequential run is a special structure containing the information about as causal dependencies of events in the present computation as the order in which they have happened.

Definition 4.1 *A sequential run of a net N is a pair (π, σ), where:*

- *a process $\pi \in \Pi(N)$ contains the information about causal dependencies of transitions which brought to this state;*

- *a sequence $\sigma \in T_C^*$ s.t. $\pi_N \xrightarrow{\sigma} \pi$, contains the information about the order in which the transitions occur which brought to this state.*

Let us denote the *set of all sequential runs* of a net N by $Runs(N)$.

The *initial* sequential run of a net N is a pair (π_N, ε), where ε is an empty sequence. Let us denote by $|\sigma|$ a *length* of a sequence σ.

Let $(\pi, \sigma), (\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$. We write $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$, $\exists \hat{\sigma} \in T_{\widetilde{C}}^*$ $\pi \xrightarrow{\hat{\sigma}} \tilde{\pi}$ and $\tilde{\sigma} = \sigma \hat{\sigma}$. We write $(\pi, \sigma) \to (\tilde{\pi}, \tilde{\sigma})$, if $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$ for some $\hat{\pi}$.

Let $(\pi, \sigma) \in Runs(N)$, $(\pi', \sigma') \in Runs(N')$ and $\sigma = v_1 \cdots v_n$, $\sigma' = v_1' \cdots v_n'$. Let us define a mapping $\beta_\sigma^{\sigma'} : T_C \to T_{C'}$ as follows: $\beta_\sigma^{\sigma'} = \{(v_i, v_i') \mid 1 \le i \le n\}$. Let $\beta_\varepsilon^\varepsilon = \emptyset$.

Let $(\pi, \sigma) \in Runs(N)$ and $\sigma = v_1 \cdots v_n$, $\pi_N \xrightarrow{v_1} \ldots \xrightarrow{v_i} \pi_i$ $(1 \le i \le n)$.

Let us introduce the following notations:

- $\pi(0) = \pi_N$,

  $\pi(i) = \pi_i$ $(1 \le i \le n)$;

- $\sigma(0) = \varepsilon$,

  $\sigma(i) = v_1 \cdots v_i$ $(1 \le i \le n)$.

4.2 Definitions of back-forth τ-bisimulation equivalences

Now we are ready to present definitions of back-forth τ-bisimulation equivalences.

Definition 4.2 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq Runs(N) \times Runs(N')$ is a $\star$-back $\star\star$-forth τ-bisimulation between N and N',*
$\star, \star\star \in \{$interleaving, step, partial word, pomset$\}$, denoted by $\mathcal{R} : N \underline{\leftrightarrow}_{\star b \star\star f}^\tau N', \star, \star\star \in \{i, s, pw, pom\}$, if:

1. *$((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.*

2. *$((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$*

    - *(back)*
      $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$,
      (a) $|vis(T_{\widehat{C}})| = 1$, if $\star = i$;
      (b) $vis(\prec_{\widehat{C}}) = \emptyset$, if $\star = s$;

      $\Rightarrow \exists (\tilde{\pi}', \tilde{\sigma}') : (\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma'), ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ and
      (a) $vis(\rho_{\widehat{C'}}) \sqsubseteq vis(\rho_{\widehat{C}})$, if $\star = pw$;
      (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C'}})$, if $\star \in \{i, s, pom\}$;
    - *(forth)*
      $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$,
      (a) $|vis(T_{\widehat{C}})| = 1$, if $\star\star = i$;
      (b) $vis(\prec_{\widehat{C}}) = \emptyset$, if $\star\star = s$;

$$\Rightarrow \exists(\tilde{\pi}', \tilde{\sigma}'): \ (\pi', \sigma') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \tilde{\sigma}'), \ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R} \ and$$

(a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, if $\star\star = pw$;

(b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$, if $\star\star \in \{i, s, pom\}$.

3. *As item 2, but the roles of N and N' are reversed.*

Two nets N and N' are $\star$-back $\star\star$-forth τ-bisimulation equivalent, $\star, \star\star \in \{$interleaving, step, partial word, pomset$\}$, denoted by $N \underline{\leftrightarrow}_{\star b \star\star f}^{\tau} N'$, if $\exists \mathcal{R}: N \underline{\leftrightarrow}_{\star b \star\star f}^{\tau} N'$, $\star, \star\star \in \{i, s, pw, pom\}$.

Let us note that back extensions of sequential runs are *deterministic*, i.e. for $(\pi, \sigma) \in Runs(N)$ there exists only one $(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$ s.t. $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$ and $|\tilde{\sigma}| = i$ $(0 \le i \le |\sigma|)$. In such a case $(\tilde{\pi}, \tilde{\sigma}) = (\pi(i), \sigma(i))$.

4.3 Interrelations of back-forth τ-bisimulation equivalences

Let us compare back-forth τ-bisimulation equivalences.

Proposition 4.1 *Let $\star \in \{i, s, pw, pom\}$. For nets N and N' $N \underline{\leftrightarrow}_{pwb\star f}^{\tau} N' \ \Leftrightarrow \ N \underline{\leftrightarrow}_{pomb\star f}^{\tau} N'$.*

Proof. ($\Leftarrow$) Isomorphism of lposets is homomorphism.
($\Rightarrow$) Let $\mathcal{R}: N \underline{\leftrightarrow}_{pwb\star f}^{\tau} N'$. Let us prove $\mathcal{R}: N \underline{\leftrightarrow}_{pomb\star f}^{\tau} N'$.

1. Obviously, $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.

2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$.

   - (back)

     Let $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$. Then $\exists(\tilde{\pi}', \tilde{\sigma}'): \ (\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma')$, $((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ and $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$.

     Due to the symmetry of a bisimulation, the back extension $(\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma')$ must be imitated by some extension $(\tilde{\tilde{\pi}}, \tilde{\tilde{\sigma}}) \xrightarrow{\check{\pi}} (\pi, \sigma)$ s.t. $vis(\rho_{\check{C}}) \sqsubseteq vis(\rho_{\widehat{C}'})$. Due to determinism of back extensions, $vis(T_{\widehat{C}}) = vis(T_{\check{C}})$. Then $vis(\rho_{\widehat{C}}) = vis(\rho_{\check{C}})$, and we obtain $vis(\rho_{\widehat{C}}) \sqsubseteq vis(\rho_{\widehat{C}'})$. Consequently, $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$.

   - (forth)
     Obviously.

3. As item 2, but the roles of N and N' are reversed. $\qquad\qquad\qquad\square$

Proposition 4.2 *Let $\star \in \{i, s, pw, pom\}$. For nets N and N' $N \underline{\leftrightarrow}_{\star bif}^{\tau} N' \ \Leftrightarrow \ N \underline{\leftrightarrow}_{\star b \star f}^{\tau} N'$.*

Proof. ($\Leftarrow$) Isomorphism of causal nets, isomorphism and homomorphism of lposets of causal nets, isomorphism of lposets of causal nets with empty precedence relation imply label preserving bijection of lposets of causal nets.
($\Rightarrow$) Let $\mathcal{R}: N \underline{\leftrightarrow}_{\star bif}^{\tau} N'$. Let us prove $\mathcal{R}: N \underline{\leftrightarrow}_{\star b \star f}^{\tau} N'$.

1. Obviously, $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.

2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$.

   - (back)
     Obviously.

   - (forth)

     Let $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$. The extension by $\hat{\pi}$ corresponds to the extension by some sequence of transitions.

     Then $\exists(\tilde{\pi}', \tilde{\sigma}'): \ (\pi', \sigma') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \tilde{\sigma}')$, $((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$, where the extension by $\hat{\pi}'$ corresponds to the extension by sequence of transitions which imitates the corresponding one in the net N.

     The back extension $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$ must be imitated by some extension $(\bar{\pi}', \bar{\sigma}') \xrightarrow{\check{\pi}'} (\tilde{\pi}', \tilde{\sigma}')$, s.t.

     (a) $vis(\rho_{\check{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, if $\star = pw$;

     (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\check{C}'})$, if $\star \in \{i, s, pom\}$.
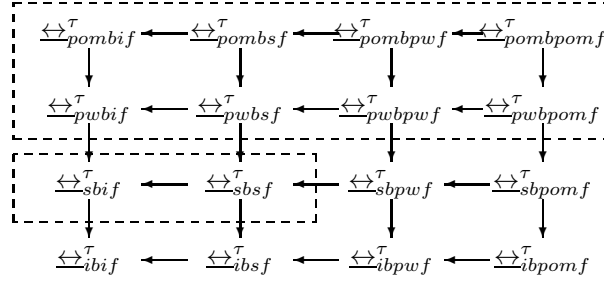
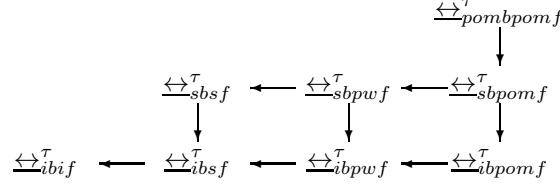Figure 6: Merging of back-forth τ-bisimulation equivalences

Figure 7: Interrelations of back-forth τ-bisimulation equivalences

Due to determinism of back extensions, $vis(T_{\widehat{C}'}) = vis(T_{\check{C}'})$. Then $vis(\rho_{\widehat{C}'}) = vis(\rho_{\check{C}'})$. Consequently,

(a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, if $\star = pw$;

(b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$, if $\star \in \{i, s, pom\}$.

3. As item 2, but the roles of N and N' are reversed. $\square$

In Figure 6, dashed lines embrace coinciding back-forth τ-bisimulation equivalences.

Hence, interrelations of back-forth τ-bisimulation equivalences may be represented by graph in Figure 7.

4.4 Interrelations of back-forth τ-bisimulation equivalences with basic τ-equivalences

Let us consider compare back-forth τ-bisimulation equivalences with basic τ-equivalences.

For some net N and $(\pi, \sigma), (\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$ we write $(\pi, \sigma) \Rightarrow (\tilde{\pi}, \tilde{\sigma})$ when $(\pi, \sigma) \rightarrow (\tilde{\pi}, \tilde{\sigma})$ and $\pi \Rightarrow \tilde{\pi}$.

Let for some nets N and N' $(\pi, \sigma) \in Runs(N)$, $(\pi', \sigma') \in Runs(N')$ and $(\pi_E, \pi_P) \in ST^\tau - \Pi(N)$, $(\pi'_E, \pi'_P) \in ST^\tau - \Pi(N')$.

We shall use the following notations.

- $(\pi, \sigma) \underline{\leftrightarrow}^\tau_{ibif}(\pi', \sigma')$ if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{ibif} N'$ s.t. $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$ and analogously for $\underline{\leftrightarrow}^\tau_{pombpomf}$.

- $\pi \underline{\leftrightarrow}_{ibr} \pi'$ if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{ibr} N'$ s.t. $(\pi, \pi') \in \mathcal{R}$.

- $\pi \underline{\leftrightarrow}^\tau_{pomhbr} \pi'$ if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhbr} N'$ $\exists \beta$ s.t. $(\pi, \pi', \beta) \in \mathcal{R}$.

- $(\pi_E, \pi_P) \underline{\leftrightarrow}^\tau_{iSTbr}(\pi'_E, \pi'_P)$ if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{iSTbr} N'$ $\exists \beta$ s.t. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$ and analogously for $\underline{\leftrightarrow}^\tau_{pomhSTbr}$.

Lemma 4.1 (X-Lemma 1) *Let for nets N and N' $N \underline{\leftrightarrow}^\tau_{ibif} N'$ and $(\pi, \sigma), (\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$, $(\pi', \sigma'), (\tilde{\pi}', \tilde{\sigma}') \in Runs(N')$ s.t. $(\pi, \sigma) \Rightarrow (\tilde{\pi}, \tilde{\sigma})$, $(\pi', \sigma') \Rightarrow (\tilde{\pi}', \tilde{\sigma}')$. Then $(\pi, \sigma) \underline{\leftrightarrow}^\tau_{ibif}(\tilde{\pi}', \tilde{\sigma}')$ and $(\tilde{\pi}, \tilde{\sigma}) \underline{\leftrightarrow}^\tau_{ibif}(\pi', \sigma')$ implies $(\pi, \sigma) \underline{\leftrightarrow}^\tau_{ibif}(\pi', \sigma')$ and $(\tilde{\pi}, \tilde{\sigma}) \underline{\leftrightarrow}^\tau_{ibif}(\tilde{\pi}', \tilde{\sigma}')$.*

Proof. As proof of the following Lemma 4.2, but using process extensions by one action only. $\square$

Lemma 4.2 (X-Lemma 2) *Let for nets N and N' $N \underline{\leftrightarrow}^\tau_{pombpomf} N'$ and $(\pi, \sigma), (\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$, (π', σ'), $(\tilde{\pi}', \tilde{\sigma}') \in Runs(N')$ s.t. $(\pi, \sigma) \Rightarrow (\tilde{\pi}, \tilde{\sigma})$, $(\pi', \sigma') \Rightarrow (\tilde{\pi}', \tilde{\sigma}')$. Then $(\pi, \sigma) \underline{\leftrightarrow}^\tau_{pombpomf}(\tilde{\pi}', \tilde{\sigma}')$ and $(\tilde{\pi}, \tilde{\sigma}) \underline{\leftrightarrow}^\tau_{pombpomf}$ (π', σ') implies $(\pi, \sigma) \underline{\leftrightarrow}^\tau_{pombpomf}(\pi', \sigma')$ and $(\tilde{\pi}, \tilde{\sigma}) \underline{\leftrightarrow}^\tau_{pombpomf}(\tilde{\pi}', \tilde{\sigma}')$.*

Proof. It is enough to prove $(\tilde{\pi}, \tilde{\sigma}) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\tilde{\pi}', \tilde{\sigma}')$, since the fact $(\pi, \sigma) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi', \sigma')$ is proved similarly. Let $(\pi, \sigma) \Rightarrow (\tilde{\pi}, \tilde{\sigma})$, $(\pi', \sigma') \Rightarrow (\tilde{\pi}', \tilde{\sigma}')$ and $(\pi, \sigma) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\tilde{\pi}', \tilde{\sigma}')$, $(\tilde{\pi}, \tilde{\sigma}) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi', \sigma')$. We have only to check similation of the net N by N' in back and forth directions, since simulation of N' by N is proved by symmetry.

- (back)

  Let $(\bar{\pi}, \bar{\sigma}) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$, $\hat{\pi} = (\widehat{C}, \hat{\varphi})$. Then, since $(\tilde{\pi}, \tilde{\sigma}) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi', \sigma')$, $\exists \check{\pi}' = (\check{C}', \check{\varphi}'), (\bar{\pi}', \bar{\sigma}')$ s.t. $(\bar{\pi}', \bar{\sigma}') \xrightarrow{\check{\pi}'} (\pi', \sigma')$, $(\bar{\pi}, \bar{\sigma}) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\bar{\pi}', \bar{\sigma}')$ and $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\check{C}'})$.

  Let us note if $(\bar{\pi}', \bar{\sigma}') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \tilde{\sigma}')$, $\hat{\pi}' = (\widehat{C}', \hat{\varphi}')$ then we have $vis(\rho_{\check{C}'}) = vis(\rho_{\widehat{C}'})$. Consequently, $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$.

- (forth)

  Let $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\bar{\pi}, \bar{\sigma})$, $\hat{\pi} = (\widehat{C}, \hat{\varphi})$. Let us note if $(\pi, \sigma) \xrightarrow{\check{\pi}} (\bar{\pi}, \bar{\sigma})$, $\check{\pi} = (\check{C}, \check{\varphi})$ then we have $vis(\rho_{\widehat{C}}) = vis(\rho_{\check{C}})$.

  Since $(\pi, \sigma) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\tilde{\pi}', \tilde{\sigma}')$, $\exists \hat{\pi}' = (\widehat{C}', \hat{\varphi}'), (\bar{\pi}', \bar{\sigma}')$ s.t. $(\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\bar{\pi}', \bar{\sigma}')$, $(\bar{\pi}, \bar{\sigma}) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\bar{\pi}', \bar{\sigma}')$ and $vis(\rho_{\check{C}}) \simeq vis(\rho_{\widehat{C}'})$. Consequently, $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$. $\square$

Proposition 4.3 *For nets N and N' $N \underline{\leftrightarrow}^{\tau}_{ibif} N' \Leftrightarrow N \underline{\leftrightarrow}^{\tau}_{ibr} N'$.*

Proof. As proof of the following Proposition 4.4, but using process extensions by one action only and Lemma 4.1. $\square$

Proposition 4.4 *For nets N and N' $N \underline{\leftrightarrow}^{\tau}_{pombpomf} N' \Leftrightarrow N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$.*

Proof. See Appendix A. $\square$

Proposition 4.5 *For nets N and N' $N \underline{\leftrightarrow}^{\tau}_{iSTbr} N' \Rightarrow N \underline{\leftrightarrow}^{\tau}_{ibsf} N'$.*

Proof. For $\pi \in \Pi(N)$ we denote $[\pi] = \{\bar{\pi} \mid \bar{\pi} \in \Pi(N), (\pi, \pi) \underline{\leftrightarrow}^{\tau}_{iSTbr} (\bar{\pi}, \bar{\pi})\}$. Let $(\pi, \sigma) \in Runs(N)$ and $\sigma = v_1 \cdots, v_n$. A *trace* of (π, σ) is defined by $trace(\pi, \sigma) = [\pi_N] l_C(v_1) [\pi(1)] \cdots [\pi(n-1)] l_C(v_n) [\pi(n)]$. A *trace modulo stuttering* of (π, σ), denoted by $stutt(\pi, \sigma)$, is obtained from $trace(\pi, \sigma)$ by replacing all triples of a kind $R\tau R$ by R.
Let $N \underline{\leftrightarrow}^{\tau}_{iSTbr} N'$, $(\pi, \sigma) \in Runs(N)$, $(\pi', \sigma') \in Runs(N')$ and $stutt(\pi, \sigma) = R_1 a_1 R_2 \cdots R_{n-1} a_n R_n$, $stutt(\pi', \sigma') = R'_1 a'_1 R'_2 \cdots R'_{m-1} a'_m R'_m$. We say that $stutt(\pi, \sigma)$ and $stutt(\pi', \sigma')$ are *isomorphic*, denoted by $stutt(\pi, \sigma) \simeq stutt(\pi', \sigma')$, if:

1. $n = m$;

2. $\forall i \ (1 \le i \le n) \ a_i = a'_i$;

3. $\forall i \ (1 \le i \le n)$ and $\pi_i \in R_i$, $\pi'_i \in R'_i$: $(\pi_i, \pi_i) \underline{\leftrightarrow}^{\tau}_{iSTbr} (\pi'_i, \pi'_i)$.

Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{((\pi, \sigma), (\pi', \sigma')) \mid (\pi, \sigma) \in Runs(N), (\pi', \sigma') \in Runs(N'), stutt(\pi, \sigma) \simeq stutt(\pi', \sigma')\}$. Let us prove $\mathcal{S} : N \underline{\leftrightarrow}^{\tau}_{ibsf} N'$.

1. $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{S}$, since $\pi_N \underline{\leftrightarrow}^{\tau}_{pomhbr} \pi_{N'}$.

2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{S}$.

   - (back)

     Let $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$ and $|vis(T_{\widehat{C}})| = 1$. Then $\exists i \ (1 \le i \le n) \ (\tilde{\pi}, \tilde{\sigma}) \in R_i$ from $trace(\pi, \sigma)$. Since $stutt(\pi, \sigma) \simeq stutt(\pi', \sigma')$, then $\exists k \ (1 \le k \le n)$ s.t. R_i corresponds to R'_k from $trace(\pi', \sigma')$. Then $\tilde{\pi} \underline{\leftrightarrow}^{\tau}_{iSTbr} \pi'(k)$. Consequently, $((\tilde{\pi}, \tilde{\sigma}), (\pi'(k), \sigma'(k))) \in \mathcal{S}$. Let us consider the back extension $(\pi'(k), \sigma'(k)) \xrightarrow{\hat{\pi}'} (\pi', \sigma')$. We have $|vis(T_{\widehat{C}'})| = 1$ and $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$.
   - (forth) Obviously, since $\underline{\leftrightarrow}^{\tau}_{iST}$ implies $\underline{\leftrightarrow}^{\tau}_s$.

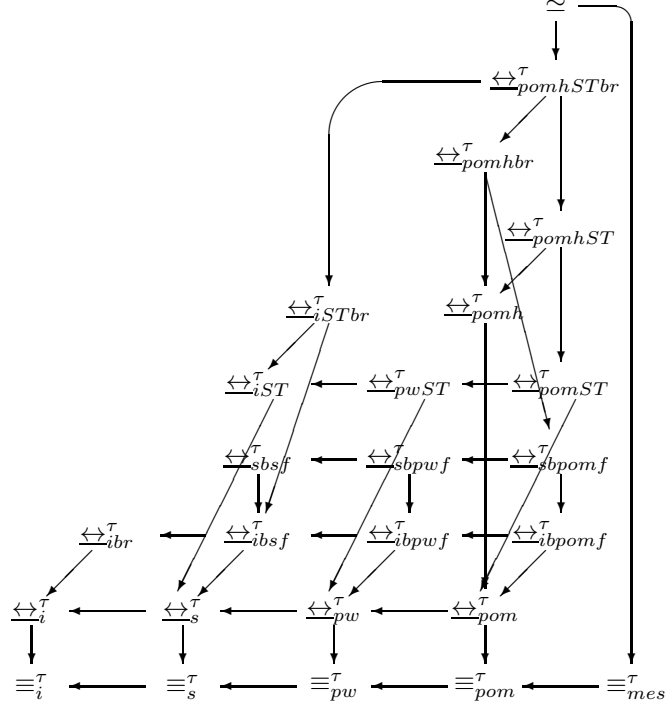3. As item 2, but the roles of N and N' are reversed. $\square$

Figure 8: Interrelations of back-forth τ-bisimulation equivalences with basic τ-equivalences

Theorem 4.1 *Let* $\leftrightarrow, \leftrightarrow\!\!\!\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau, \simeq\}$ *and* $\star, \star\star \in \{\_, i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr,$ $pomhbr, iSTbr, pomhSTbr, mes, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$. *For nets* N *and* N' $N \leftrightarrow_\star N'$ $\Rightarrow$ $N \leftrightarrow\!\!\!\leftrightarrow_{\star\star} N'$ *iff in the graph in Figure 8 there exists a directed path from* $\leftrightarrow_\star$ *to* $\leftrightarrow\!\!\!\leftrightarrow_{\star\star}$.

Proof. ($\Leftarrow$) A consequence of Theorem 3.1 and the following substantiations.

- The implication $\underline{\leftrightarrow}^\tau_{ibsf} \to \underline{\leftrightarrow}^\tau_{ibr}$ is valid since by Proposition 4.3 $\underline{\leftrightarrow}^\tau_{ibr} = \underline{\leftrightarrow}^\tau_{ibif}$ and isomorphism of lposets with empty precedence relation is isomorphism of singleton ones.

- The implications $\underline{\leftrightarrow}^\tau_{\star bpwf} \to \underline{\leftrightarrow}^\tau_{\star bsf}$, $\star \in \{i, s\}$ is valid since homomorphism is isomorphism of lposets with empty precedence relation.

- The implications $\underline{\leftrightarrow}^\tau_{\star bpomf} \to \underline{\leftrightarrow}^\tau_{\star bpwf}$, $\star \in \{i, s\}$ is valid since isomorphism of lposets is homomorphism.

- The implications $\underline{\leftrightarrow}^\tau_{ib\star f} \to \underline{\leftrightarrow}^\tau_\star$, $\star \in \{s, pw, pom\}$ is proved with constructing on the basis of the relation $\mathcal{R} : N\underline{\leftrightarrow}^\tau_{sb\star f}N'$ the new relation $\mathcal{S} : N\underline{\leftrightarrow}^\tau_\star N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \sigma, \sigma'\ ((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}\}$.

- The implications $\underline{\leftrightarrow}^\tau_{sb\star f} \to \underline{\leftrightarrow}^\tau_{ib\star f}$, $\star \in \{s, pw, pom\}$ are valid since isomorphism of lposets with empty precedence relation is isomorphism of singleton ones.

- The implication $\underline{\leftrightarrow}^\tau_{pomhbr} \to \underline{\leftrightarrow}^\tau_{sbpomf}$ is valid since by Proposition 4.4 $\underline{\leftrightarrow}^\tau_{pomhbr} = \underline{\leftrightarrow}^\tau_{pombpomf}$ and homomorphism is isomorphism of lposets with empty precedence relation.

- The implication $\underline{\leftrightarrow}^\tau_{iSTbr} \to \underline{\leftrightarrow}^\tau_{ibsf}$ is valid by Proposition 4.5.

($\Rightarrow$) An absence of additional nontrivial arrows in the graph in Figure 8 is proved by the following examples.

- In Figure 3(c), $N\underline{\leftrightarrow}^\tau_{sbsf}N'$, but $N \not\equiv^\tau_{pw} N'$.

- In Figure 9, $N\underline{\leftrightarrow}^\tau_{sbpwf}N'$, but $N \not\equiv^\tau_{pom} N'$.

- In Figure 4(a), $N\underline{\leftrightarrow}^\tau_{ibpomf}N'$, but $N\underline{\not\leftrightarrow}^\tau_{sbsf}N'$, since only in the net N' an action sequence ab can happen s.t. b should depend on a.

- In Figure 3(b), $N\underline{\leftrightarrow}^\tau_{iSTbr}N'$, but $N\underline{\not\leftrightarrow}^\tau_{sbsf}N'$ since only in the net N' an action sequence ab can happen s.t. b should depend on a. $\square$
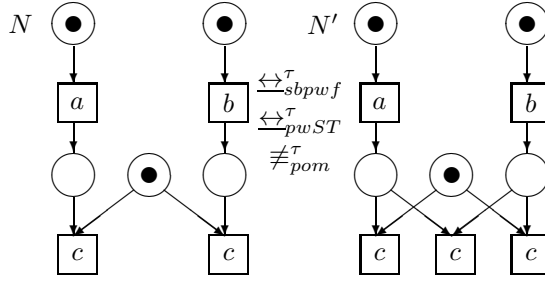
Figure 9: Example of back-forth τ-bisimulation equivalences

Thus, we obtained several important results concerning interrelations of back-forth and basic relations.

First, we have coincidences $\underleftrightarrow{}_{ibif}^{\tau} = \underleftrightarrow{}_{ibr}^{\tau}$ and $\underleftrightarrow{}_{pombpomf}^{\tau} = \underleftrightarrow{}_{pomhbr}^{\tau}$ providing branching characterization of back-forth simulation.

The second interesting result is that $\underleftrightarrow{}_{iST}^{\tau}$ implies only $\underleftrightarrow{}_{ibsf}^{\tau}$, not $\underleftrightarrow{}_{sbsf}^{\tau}$. Hence, $\underleftrightarrow{}_{iST}^{\tau}$ is not strong enough to provide step back simulation. This situation is unlike that on Petri nets without silent transitions.

4.5 Logical characterization

In this subsection, we demonstrate that several important back-forth (and branching) bisimulation equivalences coincide with that of of temporal logics having past modalities. These results provide a logical characterization of bisimulation equivalences (or, symmetrically, an operational characterization of logical ones).

4.5.1 Logic BFL

A back-forth logic (BFL) has been proposed in [15] in the framework of transition systems for a logical description of the interleaving back interleaving forth bisimulation equivalence.

Definition 4.3 *Let the symbol $\top$ denotes the truth and $a \in Act$. A* formula *of BFL is defined as follows:*

$$\Phi ::= \ \top \mid \neg\Phi \mid \Phi \wedge \Psi \mid \langle\leftarrow a\rangle\Phi \mid \langle a\rangle\Phi$$

We define $[a]\Phi = \neg\langle a\rangle\neg\Phi$ and $[\leftarrow a]\Phi = \neg\langle\leftarrow a\rangle\neg\Phi$.
We write **BFL** for the *set of all formulas* of BFL.

Definition 4.4 *Let N be some net and $(\pi, \sigma) \in Runs(N)$. The* satisfaction relation $\models_N \in Runs(N) \times \textbf{BFL}$ *is defined as follows:*

1. *$(\pi, \sigma) \models_N \top$ — always;*

2. *$(\pi, \sigma) \models_N \neg\Phi$, if $(\pi, \sigma) \not\models_N \Phi$;*

3. *$(\pi, \sigma) \models_N \Phi \wedge \Psi$, if $(\pi, \sigma) \models_N \Phi$ and $(\pi, \sigma) \models_N \Psi$;*

4. *$(\pi, \sigma) \models_N \langle\leftarrow a\rangle\Phi$, if $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$ $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$, where $\hat{\pi} = (\widehat{C}, \hat{\varphi})$, $vis(l_{\widehat{C}}(T_{\widehat{C}})) = a$ and $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$;*

5. *$(\pi, \sigma) \models_N \langle a\rangle\Phi$, if $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$ $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$, where $\hat{\pi} = (\widehat{C}, \hat{\varphi})$, $vis(l_{\widehat{C}}(T_{\widehat{C}})) = a$ and $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$.*

Definition 4.5 *We write $N \models_N \Phi$, if $(\pi_N, \varepsilon) \models_N \Phi$. Two nets N and N' are* logical equivalent *in BFL, denoted by $N =_{BFL} N'$, if $\forall\Phi \in \textbf{BFL}$ $N \models_N \Phi \ \Leftrightarrow \ N' \models_{N'} \Phi$.*

Let N be a net and $\pi \in \Pi(N)$, $a \in Act$. The set of *visible extensions* of a process π by an action a *(image set)* is defined as follows: $VisImage(\pi, a) = \{\tilde{\pi} \mid \pi \xrightarrow{\hat{\pi}} \tilde{\pi}, \ \hat{\pi} = (\widehat{C}, \hat{\varphi}), \ vis(l_{\widehat{C}}(T_{\widehat{C}})) = a\}$. A net N is a *finite-image* one, if $\forall\pi \in \Pi(N) \ \forall a \in Act \ |VisImage(\pi, a)| < \infty$.

Theorem 4.2 *[15] For two image-finite nets N and N' $N\underleftrightarrow{}_{ibr}^{\tau}N' \ \Leftrightarrow \ N\underleftrightarrow{}_{ibif}^{\tau}N' \ \Leftrightarrow \ N =_{BFL} N'$.*

In Figure 5(c), $N \underline{\leftrightarrow}^\tau_{pomhST} N'$, but $N \neq_{BFL} N'$, because for $\Phi = \langle a \rangle [\leftarrow a] \langle b \rangle \top$ $N \not\models_N \Phi$, but $N' \models_{N'} \Phi$, since in the net N' an action a can happen so that it will be simulated by sequence of actions τa in N. Then the state of the net N reached after τ must be related with the initial state of a net N, but in such a case the occurrence of action b from the initial state of N' cannot be imitated from the corresponding state of N.

Thus, in interleaving semantics, we obtained a logical characterization of branching and back-forth relations or, symmetrically, an operational characterization of equivalence imposed by back-forth logic.

4.5.2 Logic $SPBFL$

A pomset back-forth logic with invisible actions ($SPBFL$) has been proposed in [19] in the framework of event structures for a logical description of the pomset back pomset forth bisimulation equivalence.

Definition 4.6 *Let the symbol $\top$ denotes the truth and ρ be a pomset with labeling into Act. A formula of $SPBFL$ is defined as follows:*

$$\Phi ::= \ \top \ | \ \neg \Phi \ | \ \Phi \wedge \Psi \ | \ \langle \leftarrow \rho \rangle \Phi \ | \ \langle a \rangle \Phi$$

We define $[a]\Phi = \neg \langle a \rangle \neg \Phi$ and $[\leftarrow \rho]\Phi = \neg \langle \leftarrow \rho \rangle \neg \Phi$.

We write **SPBFL** for the *set of all formulas of $SPBFL$*.

Let us note that in the formula $\langle a \rangle \Phi$, corresponding to the case of forth extension, we use an action a, not a pomset ρ, since $\underline{\leftrightarrow}_{pombif} = \underline{\leftrightarrow}_{pombpomf}$. Hence, it is sufficient to consider forth extensions by one action only.

Definition 4.7 *Let N be some net and $(\pi, \sigma) \in Runs(N)$. The satisfaction relation $\models_N \in Runs(N) \times$ **SPBFL** is defined as follows:*

1. $(\pi, \sigma) \models_N \top$ — *always;*

2. $(\pi, \sigma) \models_N \neg \Phi$, *if $(\pi, \sigma) \not\models_N \Phi$;*

3. $(\pi, \sigma) \models_N \Phi \wedge \Psi$, *if $(\pi, \sigma) \models_N \Phi$ and $(\pi, \sigma) \models_N \Psi$;*

4. $(\pi, \sigma) \models_N \langle \leftarrow \rho \rangle \Phi$, *if $\exists (\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$ $(\tilde{\pi}, \tilde{\sigma}) \overset{\hat{\pi}}{\to} (\pi, \sigma)$, where $\hat{\pi} = (\widehat{C}, \hat{\varphi})$, $vis(\rho_{\widehat{C}}) \in \rho$ and $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$;*

5. $(\pi, \sigma) \models_N \langle a \rangle \Phi$, *if $\exists (\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$ $(\pi, \sigma) \overset{\hat{\pi}}{\to} (\tilde{\pi}, \tilde{\sigma})$, where $\hat{\pi} = (\widehat{C}, \hat{\varphi})$, $vis(l_{\widehat{C}}(T_{\widehat{C}})) = a$ and $(\tilde{\pi}, \tilde{\sigma}) \models_N$ Φ.*

Definition 4.8 *We write $N \models_N \Phi$, if $(\pi_N, \varepsilon) \models_N \Phi$. Two nets N and N' are logical equivalent in BFL, denoted by $N =_{SPBFL} N'$, if $\forall \Phi \in$ **SPBFL** $N \models_N \Phi \ \Leftrightarrow \ N' \models_{N'} \Phi$.*

Theorem 4.3 *[19] For two image-finite nets N and N' $N \underline{\leftrightarrow}^\tau_{pomhbr} N' \ \Leftrightarrow \ N \underline{\leftrightarrow}^\tau_{pombpomf} N' \ \Leftrightarrow \ N =_{SPBFL} N'$.*

In Figure 4(b), $N =_{BFL} N'$, but $N \neq_{SPBFL} N'$, because for $\Phi = [a][b]\langle c \rangle \langle \leftarrow (a; b) \| c \rangle \top$ $((a; b) \| c$ denotes the pomset where b depends on a, and a, b are independent with c), $N \models_N \Phi$, but $N' \not\models_{N'} \Phi$ since only in the net N' after action a action b can happen so that action c must depend on a.

Thus, in pomset semantics, we obtained a logical characterization of branching and back-forth relations or, symmetrically, an operational characterization of equivalence imposed by back-forth logic.

5 Interrelations of equivalences with τ-equivalences

In this section, we compare equivalences which do not abstract of silent actions with all the considered τ-equivalences.

Proposition 5.1 *Let $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$, $\star \in \{i, s, pw, pom, iST, pwST, pomST, mes, sbsf, sbpwf, sbpomf\}$, $\star\star \in \{s, pw, pom\}$. For nets N and N':*

1. $N \leftrightarrow_\star N' \ \Rightarrow \ N \leftrightarrow^\tau_\star N'$;

2. $N \underline{\leftrightarrow}_i N' \ \Rightarrow \ N \underline{\leftrightarrow}^\tau_{ibr} N'$;

3. $N \underline{\leftrightarrow}_{iST} N' \ \Rightarrow \ N \underline{\leftrightarrow}^\tau_{iSTbr} N'$;

4. $N \underline{\leftrightarrow}_{pomh} N' \ \Rightarrow \ N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$;
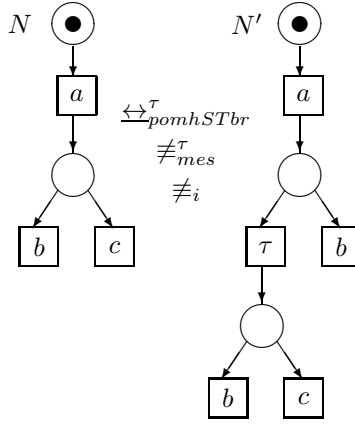
Figure 10: Example of interrelations of equivalences and τ-equivalences

5. $N\underline{\leftrightarrow}_{\star\star}N' \;\Rightarrow\; N\underline{\leftrightarrow}^{\tau}_{ib\star\star f}N'$.

and all the implications are strict.

Proof.

1. By definitions.

2. By definitions.

3. By definitions.

4. We prove with construction one the basis of the relation $\mathcal{R} : N\underline{\leftrightarrow}_{pomh}N'$ the new relation $\mathcal{S} : N\underline{\leftrightarrow}^{\tau}_{pomhSTbr}N$, defined as follows: $\mathcal{S} = \{((\pi_E,\pi_P),(\pi'_E,\pi'_P),\beta) \mid (\pi_E,\pi'_E,\beta) \in \mathcal{R},\ (\pi_E,\pi_P) \in ST^{\tau} - \Pi(N),\ (\pi'_E,\pi'_P) \in ST^{\tau} - \Pi(N'),\ \beta(T_{C_P}) = T_{C'_P}\}$.

5. We prove with construction one the basis of the relation $\mathcal{R} : N\underline{\leftrightarrow}_{\star\star}N'$ the new relation $\mathcal{S} : N\underline{\leftrightarrow}^{\tau}_{ib\star\star f}N'$, defined as follows: $\mathcal{S} = \{((\pi,\sigma),(\pi',\sigma')) \mid (\pi,\sigma) \in Runs(N),\ (\pi',\sigma') \in Runs(N'),\ |\sigma| = |\sigma'|,\ l_C(\sigma) = l_{C'}(\sigma'),\ \forall i\ (0 \le i \le |\sigma|)\ (\pi(i),\pi'(i)) \in \mathcal{R}\}$.

The strictness of the implications is proved by the following examples.

- In Figure 10, $N\underline{\leftrightarrow}^{\tau}_{pomhSTbr}N'$, but $N \not\equiv_i N'$, since only in the net N' an action a can happen in the initial state.

- In Figure 5(a), $N \equiv^{\tau}_{mes} N'$, but $N \not\equiv_i N'$, since only in the net N' an action τ can happen in the initial state. $\qquad\square$

We obtained several interesting results.

It is clear that abstraction of silent actions results weaker equivalence notions. So, implication 1 from Proposition 5.1 is rather obvious. But the other implications are not so trivial.

Implications 2–4 show that the branching idea is applicable only if to respect silent actions.

Implication 5 shows that interleaving back simulation results new equivalences only in the case of respect of silent actions.

6 Preservation of the τ-equivalences by refinements

In this section, we treat the considered τ-equivalences for preservation by transition refinements. We use SM-refinement, i.e. refinement by a special subclass of state-machine nets introduced in [5].

Definition 6.1 *An* SM-net *is a net* $D = \langle P_D, T_D, F_D, l_D, M_D \rangle$ *s.t.:*

1. $\forall t \in T_D\ |{}^{\bullet}t| = |t^{\bullet}| = 1$, i.e. each transition has exactly one input and one output place;

2. $\exists p_{in},\ p_{out} \in P_D$ s.t. $p_{in} \neq p_{out}$ and $^\circ D = \{p_{in}\}$, $D^\circ = \{p_{out}\}$, i.e. net D has unique input and unique output place.

3. $M_D = \{p_{in}\}$, i.e. at the beginning there is unique token in p_{in}.

Definition 6.2 Let $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$ be some net, $a \in l_N(T_N)$ and $D = \langle P_D, T_D, F_D, l_D, M_D \rangle$ be SM-net. An SM-refinement, denoted by $ref(N, a, D)$, is (up to isomorphism) a net $\overline{N} = \langle P_{\overline{N}}, T_{\overline{N}}, F_{\overline{N}}, l_{\overline{N}}, M_{\overline{N}} \rangle$, where:

- $P_{\overline{N}} = P_N \cup \{\langle p, u \rangle \mid p \in P_D \setminus \{p_{in}, p_{out}\},\ u \in l_N^{-1}(a)\}$;

- $T_{\overline{N}} = (T_N \setminus l_N^{-1}(a)) \cup \{\langle t, u \rangle \mid t \in T_D,\ u \in l_N^{-1}(a)\}$;

- $F_{\overline{N}}(\bar{x}, \bar{y}) = \begin{cases} F_N(\bar{x}, \bar{y}), & \bar{x}, \bar{y} \in P_N \cup (T_N \setminus l_N^{-1}(a)); \\ F_D(x, y), & \bar{x} = \langle x, u \rangle,\ \bar{y} = \langle y, u \rangle,\ u \in l_N^{-1}(a); \\ F_N(\bar{x}, u), & \bar{y} = \langle y, u \rangle,\ \bar{x} \in {}^\bullet u,\ u \in l_N^{-1}(a),\ y \in p_{in}^\bullet; \\ F_N(u, \bar{y}), & \bar{x} = \langle x, u \rangle,\ \bar{y} \in {}^\bullet u,\ u \in l_N^{-1}(a),\ x \in {}^\bullet p_{out}; \\ 0, & \text{otherwise}; \end{cases}$

- $l_{\overline{N}}(\bar{u}) = \begin{cases} l_N(\bar{u}), & \bar{u} \in T_N \setminus l_N^{-1}(a); \\ l_D(t), & \bar{u} = \langle t, u \rangle,\ t \in T_D,\ u \in l_N^{-1}(a); \end{cases}$

- $M_{\overline{N}}(p) = \begin{cases} M_N(p), & p \in P_N; \\ 0, & \text{otherwise.} \end{cases}$

An equivalence is *preserved by refinements*, if equivalent nets remain equivalent after applying any refinement operator to them accordingly.

The following proposition demonstrates that some considered in the paper equivalence notions are not preserved by SM-refinements.

Proposition 6.1 Let $\star \in \{i, s\}$, $\star\star \in \{i, s, pw, pom, pomh, ibr, pomhbr, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$. Then the τ-equivalences $\equiv_\star^\tau$, $\underline{\leftrightarrow}_{\star\star}^\tau$ are not preserved by SM-refinements.

Proof.

- In Figure 11, $N \underline{\leftrightarrow}_s^\tau N'$, but $ref(N, c, D) \not\equiv_i^\tau ref(N', c, D)$, since only in $ref(N', c, D)$ the sequence of actions $c_1 a b c_2$ can happen. Consequently, the τ-equivalences between $\equiv_i^\tau$ and $\underline{\leftrightarrow}_s^\tau$ are not preserved by SM-refinements.

- In Figure 12, $N \underline{\leftrightarrow}_{pom}^\tau N'$, but $ref(N, a, D) \not\underline{\leftrightarrow}_i^\tau ref(N', a, D)$, since only in $ref(N', a, D)$ after occurrence of action a_1 action b can not happen. Consequently, no equivalence between $\underline{\leftrightarrow}_i^\tau$ and $\underline{\leftrightarrow}_{pom}^\tau$ is preserved by SM-refinements.

- In Figure 13, $N \underline{\leftrightarrow}_{pomhbr}^\tau N'$, but $ref(N, a, D) \not\underline{\leftrightarrow}_i^\tau ref(N', a, D)$, since only in $ref(N', a, D)$ an action c_1 may happen so that after the corresponding action c_1 in the net N an action a may happen in such a way that the action b never occur. Consequently, no equivalence between $\underline{\leftrightarrow}_i^\tau$ and $\underline{\leftrightarrow}_{pomhbr}^\tau$ is preserved by SM-refinements. Let us note that this figure is a translation of an example on event structures from [19] to the framework of Petri nets.

In Figure 14, lines embrace τ-equivalences which are not preserved by SM-refinements due to examples in Figures 11–13. □

Let us consider which τ-equivalences are preserved by SM-refinements.

Proposition 6.2 Let $\star \in \{pw, pom\}$. For nets N, N' s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ and SM-net D $N \equiv_\star^\tau N' \Rightarrow ref(N, a, D) \equiv_\star^\tau ref(N', a, D)$.

Proof. See Appendix B. □

Proposition 6.3 Let $\star \in \{i, pw, pom\}$. For nets N, N' s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ and SM-net D $N \underline{\leftrightarrow}_{\star ST}^\tau N' \Rightarrow ref(N, a, D) \underline{\leftrightarrow}_{\star ST}^\tau ref(N', a, D)$.
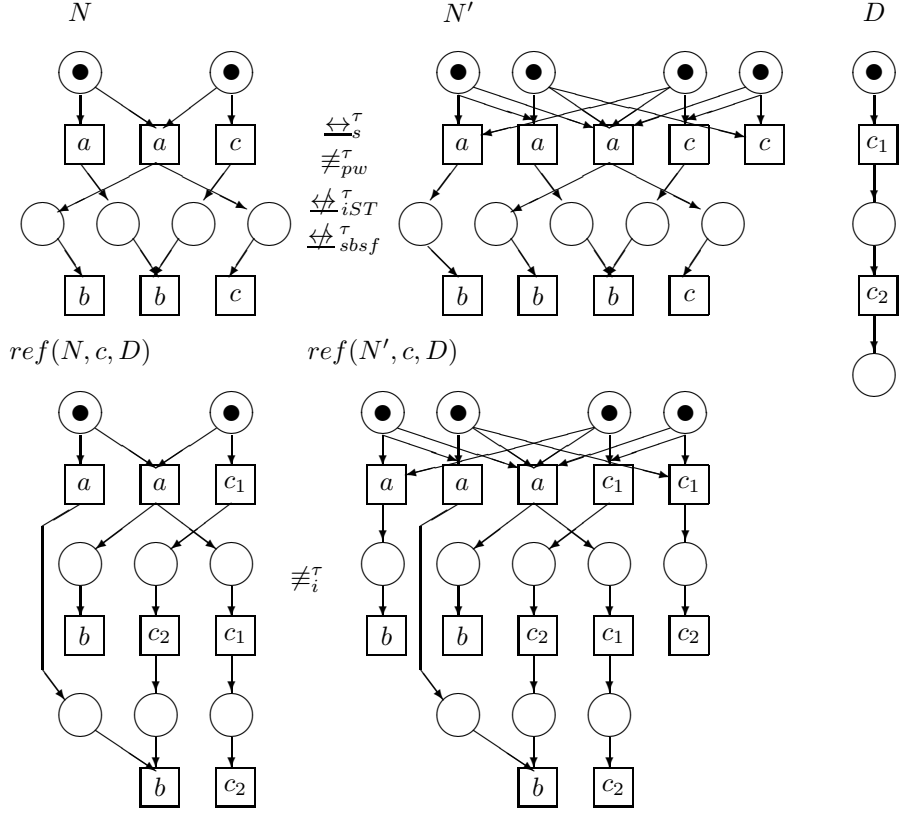
Proof. See Appendix C. □

Figure 11: The τ-equivalences between $\equiv_i^\tau$ and $\underline{\leftrightarrow}_s^\tau$ are not preserved by SM-refinements

Proposition 6.4 *[5, 10] For nets N, N' s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ and SM-net D $N \underline{\leftrightarrow}_{pomhST}^\tau N' \Rightarrow ref(N, a, D) \underline{\leftrightarrow}_{pomhST}^\tau ref(N', a, D)$.*

Proposition 6.5 *For nets N, N' s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ and SM-net D $N \underline{\leftrightarrow}_{iSTbr}^\tau N' \Rightarrow ref(N, a, D) \underline{\leftrightarrow}_{iSTbr}^\tau ref(N', a, D)$.*

Proof. Like proof of the case $\star = i$ in Proposition 6.3, but with check of branching simulation. $\qquad\square$

Proposition 6.6 *For nets N, N' s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ and SM-net D $N \underline{\leftrightarrow}_{pomhSTbr}^\tau N' \Rightarrow ref(N, a, D) \underline{\leftrightarrow}_{pomhSTbr}^\tau ref(N', a, D)$.*

Proof. Like proof of Proposition 6.4, but with check of branching simulation. $\qquad\square$

Proposition 6.7 *For nets N, N' s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ and SM-net D $N \equiv_{mes}^\tau N' \Rightarrow ref(N, a, D) \equiv_{mes}^\tau ref(N', a, D)$.*

Proof. See Appendix D. $\qquad\square$

Proposition 6.8 *For nets N, N' s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'})$ and SM-net D $N \simeq N' \Rightarrow ref(N, a, D) \simeq ref(N', a, D)$.*

Proof. Obviously. $\qquad\square$

Theorem 6.1 *Let $\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau, \simeq\}$ and $\star \in \{\_, i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr, pomhbr, iSTbr, pomhSTbr, mes, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$. For nets N, N' s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ and SM-net D the following holds: $N \leftrightarrow_\star N' \Rightarrow ref(N, a, D) \leftrightarrow_\star ref(N', a, D)$ iff the equivalence $\leftrightarrow_\star$ is in oval in Figure 15.*
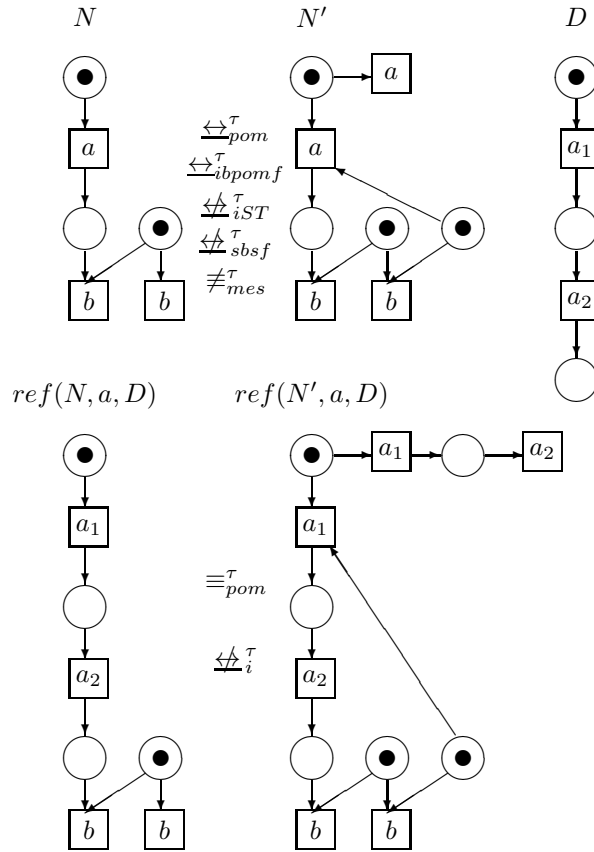
25

Figure 12: The τ-equivalences between $\underline{\leftrightarrow}_i^\tau$ and $\underline{\leftrightarrow}_{pom}^\tau$ are not preserved by SM-refinements
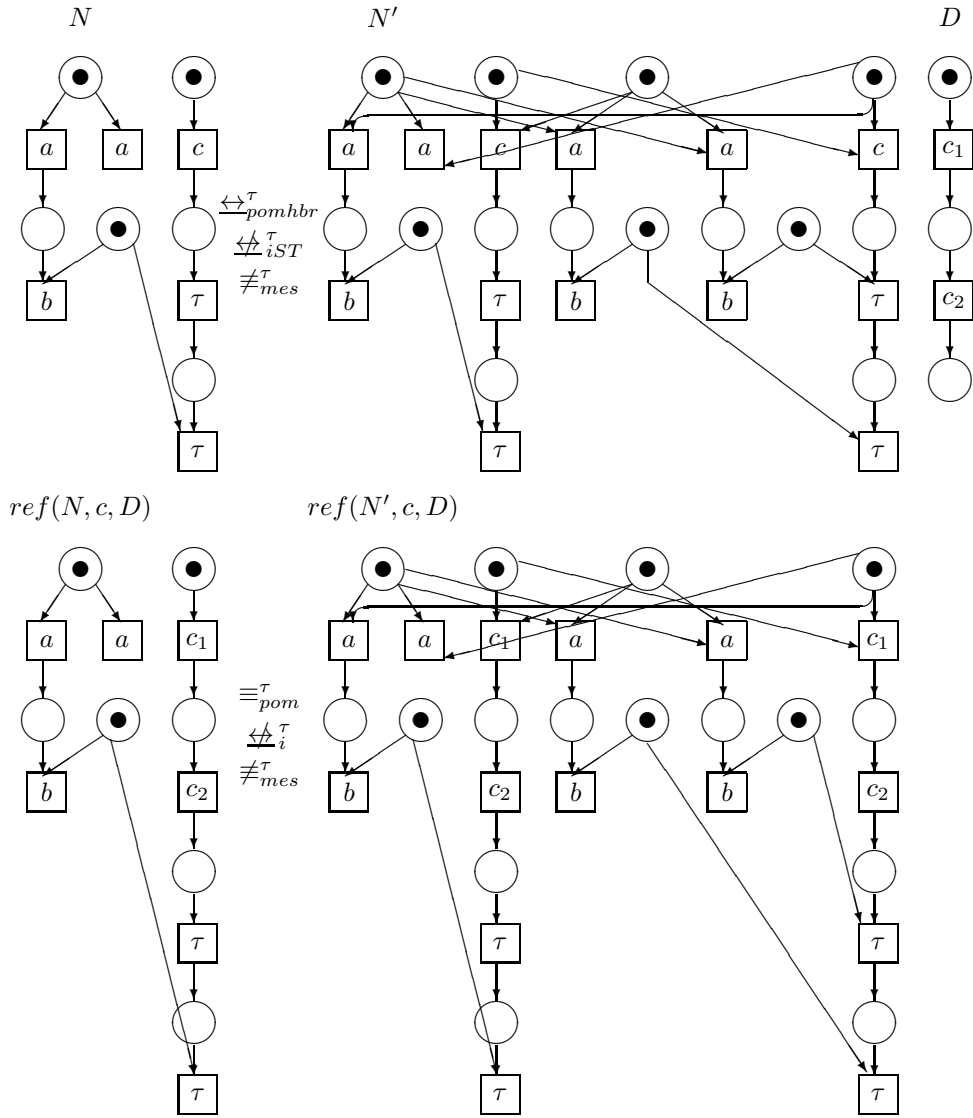
Figure 13: The τ-equivalences between $\underline{\leftrightarrow}_i^\tau$ and $\underline{\leftrightarrow}_{pomhbr}^\tau$ are not preserved by SM-refinements
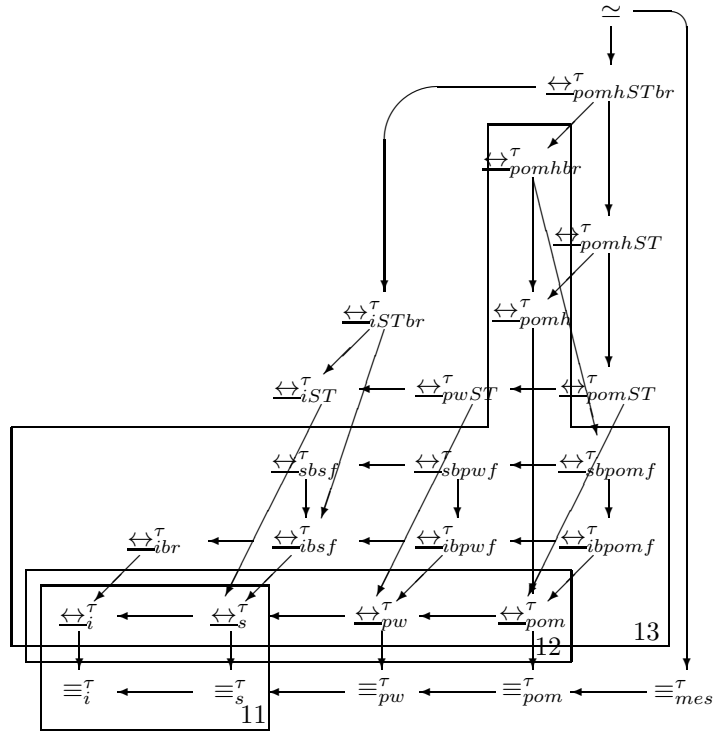
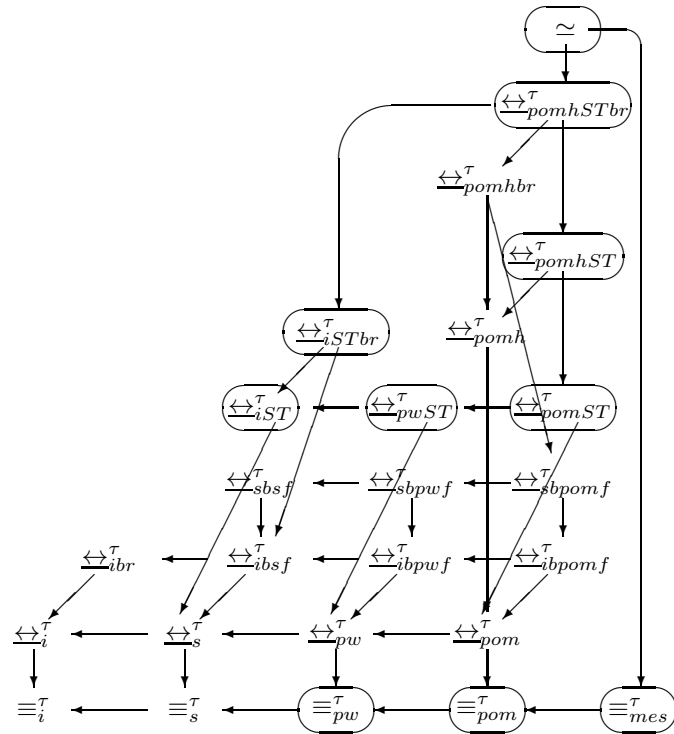Figure 14: The τ-equivalences which are not preserved by SM-refinements

Figure 15: Preservation of the τ-equivalences by SM-refinements

Proof. By Propositions 6.1–6.8. □

Thus, we obtained several interesting results concerning preservation by refinements.

First, $\equiv^\tau_{pw}$, $\equiv^\tau_{pom}$ and $\equiv^\tau_{mes}$ are preserved by this operation.

The second result is that all the ST-equivalences withstand this operation too. Our new ST-equivalences are proved to be helpful in top-down design. If one wants to have for multilevel design a notion of branching equivalence and needs only interleaving semantics, he takes $\underleftrightarrow{}^\tau_{iSTbr}$. In pomset semantics, $\underleftrightarrow{}^\tau_{pomhSTbr}$ is appropriate.

7 The τ-equivalences on some net subclasses

In this section, we consider the τ-equivalences on nets without silent transitions and sequential nets.

7.1 The τ-equivalences on nets without silent transitions

Let us consider the τ-equivalences on nets without silent transitions, where no transition is labeled by the action τ.

Proposition 7.1 *Let* $\leftrightarrow \in \{\equiv, \underleftrightarrow{}\}$, $\star \in \{i, s, pw, pom, iST, pwST, pomST, mes, sbsf, sbpwf, sbpomf\}$, $\star\star \in \{s, pw, pom\}$. *For nets without silent transitions* N *and* N':

1. $N \leftrightarrow_\star N' \Leftrightarrow N \leftrightarrow^\tau_\star N'$;

2. $N \underleftrightarrow{}_i N' \Leftrightarrow N \underleftrightarrow{}^\tau_{ibr} N'$;

3. $N \underleftrightarrow{}_{iST} N' \Leftrightarrow N \underleftrightarrow{}^\tau_{iSTbr} N'$;

4. $N \underleftrightarrow{}_{pomh} N' \Leftrightarrow N \underleftrightarrow{}^\tau_{pomhSTbr} N'$;

5. $N \underleftrightarrow{}_{\star\star} N' \Leftrightarrow N \underleftrightarrow{}^\tau_{ib\star\star f} N'$.

Proof. ($\Leftarrow$)

1. By definitions.

2. By definitions.

3. By definitions.

4. We prove with construction one the basis of the relation $\mathcal{R} : N \underleftrightarrow{}^\tau_{pomhSTbr} N'$ the new relation $\mathcal{S} : N \underleftrightarrow{}_{pomh} N$, defined as follows: $\mathcal{S} = \{(\pi, \pi', \beta) \mid ((\pi, \pi), (\pi', \pi'), \beta) \in \mathcal{R}\}$.

5. We prove with construction one the basis of the relation $\mathcal{R} : N \underleftrightarrow{}^\tau_{ib\star\star f} N'$ the new relation $\mathcal{S} : N \underleftrightarrow{}_{\star\star} N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \sigma, \sigma'$
   $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}\}$.

($\Rightarrow$) By Proposition 5.1, because nets without silent transitions are a subclass of that of with silent transitions. □

In Figure 16, dashed lines embrace the τ-equivalences coinciding on nets without silent transitions.

Theorem 7.1 *Let* $\leftrightarrow, \leftrightarrow\!\!\!\leftrightarrow \in \{\equiv, \underleftrightarrow{}, \simeq\}$, $\star, \star\star \in \{\_, i, s, pw, pom, iST, pwST, pomST, pomh, ibr, mes, sbsf, sbpwf, sbpomf\}$. *For nets without silent transitions* N *and* N' $N \leftrightarrow_\star N' \Rightarrow N \leftrightarrow\!\!\!\leftrightarrow_{\star\star} N'$ *iff in the graph in Figure 17 there exists a directed path from* $\leftrightarrow_\star$ *to* $\leftrightarrow\!\!\!\leftrightarrow_{\star\star}$.

Proof. By Proposition 7.1 and Theorem 1 from [20]. □

Thus, we have several interesting results.

It is clear that abstraction of silent actions plays no role in Petri nets without silent transitions. Hence, we obtain coincidence of relations abstracting of silent actions with that of not abstracting, and equality 1 from Proposition 7.1 is obvious. But the other equalities are not so trivial.

Equalities 2–4 show that the branching idea is applicable only if to respect silent actions.

Equality 5 shows that interleaving back simulation results new equivalences only in the case of respect of silent actions.
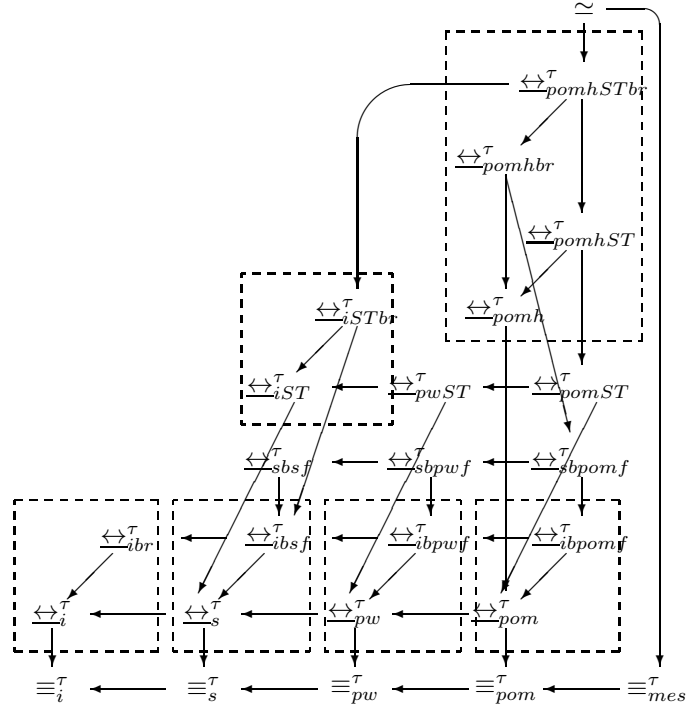
Figure 16: Merging of the τ-equivalences on nets without silent transitions
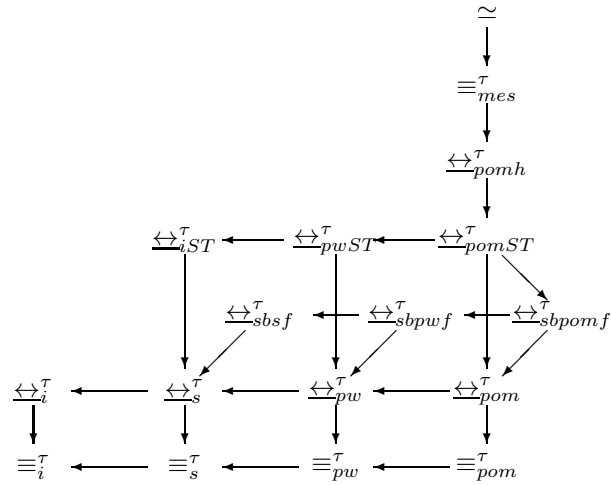

Figure 17: Interrelations of the τ-equivalences on nets without silent transitions

7.2 The τ-equivalences on sequential nets

Let us consider the τ-equivalences on sequential nets, where no two transitions can be fired concurrently.

Definition 7.1 *A net $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$ is* sequential, *if $\forall M \in Mark(N) \ \neg \exists t, u \in T_N : \ {}^\bullet t + {}^\bullet u \subseteq M$.*

Proposition 7.2 *For sequential nets N and N':*

1. $N \equiv_i^\tau N' \ \Leftrightarrow \ N \equiv_{pom}^\tau N'$;

2. $N \underline{\leftrightarrow}_i^\tau N' \ \Leftrightarrow \ N \underline{\leftrightarrow}_{pomh}^\tau N'$;

3. $N \underline{\leftrightarrow}_{iST}^\tau N' \ \Leftrightarrow \ N \underline{\leftrightarrow}_{pomhST}^\tau N'$;

4. $N \underline{\leftrightarrow}_{ibr}^\tau N' \ \Leftrightarrow \ N \underline{\leftrightarrow}_{pomhbr}^\tau N'$;

5. $N \underline{\leftrightarrow}_{iSTbr}^\tau N' \ \Leftrightarrow \ N \underline{\leftrightarrow}_{pomhSTbr}^\tau N'$.

Proof.

1. ($\Leftarrow$) By Theorem 3.1.

   ($\Rightarrow$) Let $N \equiv_i^\tau N'$, then $VisIntTraces(N) = VisIntTraces(N')$. To prove $N \equiv_{pom}^\tau N'$, it is sufficient to establish the equality $VisPomsets(N) = VisPomsets(N')$. It follows immediately, since $VisPomsets(N)$ and $VisPomsets(N')$ are totally ordered multisets (chains), and there is on-to-one correspondence between $VisIntTraces(N)$ and $VisPomsets(N)$ ($VisIntTraces(N')$ and $VisPomsets(N')$ respectively).

2. By Proposition 5.4 from [5].

3. Similar to the item 2.

4. Similar to the item 2.

5. Similar to the item 2. □

In Figure 18, dashed lines embrace the τ-equivalences coinciding on sequential nets.

Theorem 7.2 *Let $\leftrightarrow, \nleftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau, \simeq\}$, $\star, \star\star \in \{\_, i, iST, ibr, iSTbr, mes\}$. For sequential nets N and N' $N \leftrightarrow_\star N' \ \Rightarrow \ N \nleftrightarrow_{\star\star} N'$ iff in the graph in Figure 19 there exists a directed path from $\leftrightarrow_\star$ to $\nleftrightarrow_{\star\star}$.*

Proof. ($\Leftarrow$) By Proposition 7.2 and Theorem 4.1.
($\Rightarrow$) An absence of additional nontrivial arrows in the graph in Figure 19 is proved by the following examples on sequential nets.

- In Figure 5(a), $N \equiv_{mes}^\tau N'$, but $N \not\underline{\leftrightarrow}_i^\tau N'$.

- In Figure 5(c), $N \underline{\leftrightarrow}_i^\tau N'$, but $N \not\underline{\leftrightarrow}_{ibr}^\tau N'$.

- In Figure 5(b), $N \underline{\leftrightarrow}_i^\tau N'$, but $N \not\underline{\leftrightarrow}_{iST}^\tau N'$.

- In Figure 4(c), $N \underline{\leftrightarrow}_{iSTbr}^\tau N'$, but $N \not\equiv_{mes}^\tau N'$. □

Thus, we obtained several important results.

First, it is clear that on sequential nets all pomsets of processes are strictly ordered and they are simple chains. So, all interleaving and pomset equivalences coincide, and equality 1 from Proposition 7.2 is obvious. But the other equalities are not so trivial.

The basic is equality 2 showing coincidence of interleaving and pomset history preserving relations. Thus, history preservation idea on sequential nets provide no special equivalence notions.

Equalities 3–5 are sequences of 2. They additionally take into account ST-, branching ideas and both ideas together.
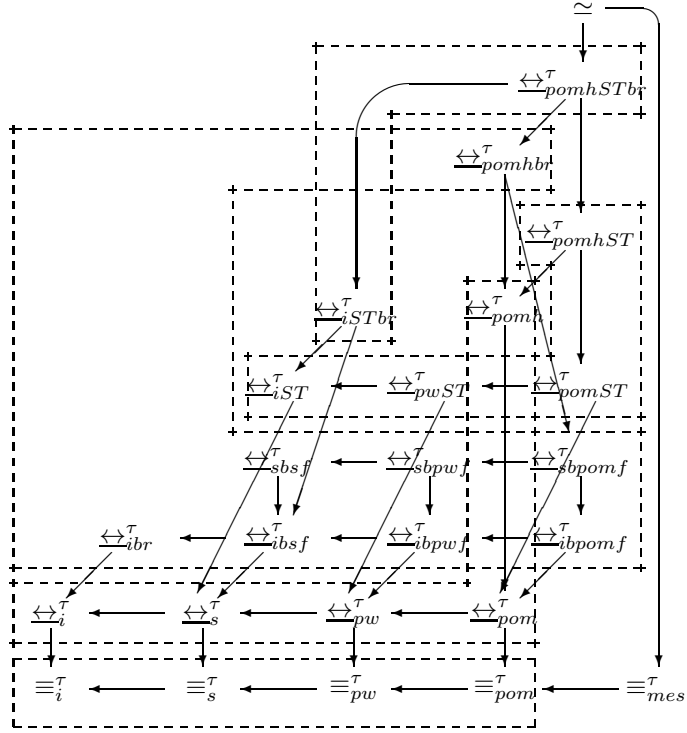
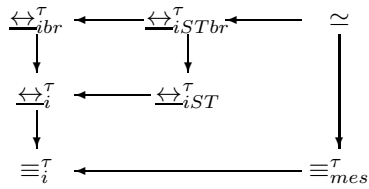Figure 18: Merging of the τ-equivalences on sequential nets

Figure 19: Interrelations of the τ-equivalences on sequential nets

$$\sim^\tau_{ibr}$$
$$\downarrow$$
$$\sim^\tau_i \longleftarrow \sim^\tau_s \longleftarrow \sim^\tau_{pw} \longleftarrow \sim^\tau_{pom}$$
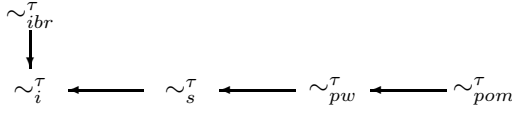
Figure 20: Interrelations of place τ-bisimulation equivalences

8   Conclusion

In this paper, we supplemented by new ones and examined a group of basic τ-equivalences and back-forth τ-bisimulation equivalences. We compared them with relations which do not abstract of silent actions. We also compared them on the whole class of Petri nets as well as on their subclasses of nets without silent transitions and sequential nets. All the considered τ-equivalences were checked for preservation by SM-refinements. So, we can use the τ-equivalence notions that are preserved by SM-refinements, for top-down design of concurrent systems.

Further research may consist in the investigation of τ-variants of place bisimulation equivalences [2] which are used for effective semantically correct reduction of nets. In [23], we have already investigated place equivalences for Petri nets without silent transitions. So, our aim is to extend these results to wider net class. In [3, 1], a notion of interleaving place τ-bisimulation equivalence ($\sim^\tau_i$) was proposed, and its usefulness for behavior preserving simplification of Petri nets with silent transitions was demonstrated. It was mentioned that τ-variants of place bisimulations provide much more reductions than usual ones because of merging many silent transitions.

In interleaving semantics, it is possible to define branching place relation ($\sim^\tau_{ibr}$) as well. It would be very interesting to treat also non-interleaving variants of place τ-bisimulations ($\sim^\tau_s, \sim^\tau_{pw}$ and $\sim^\tau_{pom}$) in order to respect true concurrency aspects during reduction of nets. Thus, we obviously have the diagram of interrelations shown in Figure 20.

A hard question here is to find whether any of three relations $\sim^\tau_i, \sim^\tau_s$ and $\sim^\tau_{pw}$ coincide like it was for the corresponding notions not abstracting of silent actions (we had coincidence of all the three analogous relations in that case). At the present moment, we have only counterexamples showing that $\sim^\tau_{ibr}$ and $\sim^\tau_{pom}$ do not imply each other and do not merge with any of three mentioned τ-equivalences. In addition, we should establish interrelations of the place notions with all τ-equivalences we proposed in this paper.

What is about preservation by SM-refinements, the results of [23] demonstrate that no place τ-bisimulation relation is preserved by the transformation.

Obviously, on Petri nets without silent transitions place τ-equivalences coincide with the corresponding relations that do not abstract of silent actions. In particular, $\sim^\tau_{ibr}$ merges with $\sim_i$. On sequential nets, all non-interleaving place relations coincide with interleaving ones. Hence, only $\sim^\tau_i$ and $\sim^\tau_{ibr}$ are remained.

Thus, we presented several ideas concerning place τ-bisimulations. We leave general research in this area for the future.

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A Proof of Proposition 4.4

Let us note that the following proof is a translation of that for event structures from [19] to the framework of Petri nets.

For $\pi \in \Pi(N)$ we denote $[\pi] = \{\bar{\pi} \mid \bar{\pi} \in \Pi(N), \ \pi \underline{\leftrightarrow}^\tau_{pomhbr} \bar{\pi}\}$. Let $(\pi, \sigma) \in Runs(N)$ and $\sigma = v_1 \cdots, v_n$. A *trace* of (π, σ) is defined by $trace(\pi, \sigma) = [\pi_N] l_C(v_1)[\pi(1)] \cdots [\pi(n-1)] l_C(v_n)[\pi(n)]$. A *trace modulo stuttering* of (π, σ), denoted by $stutt(\pi, \sigma)$, is obtained from $trace(\pi, \sigma)$ by replacing all triples of a kind $R\tau R$ by R.

($\Leftarrow$) Let $N \underline{\leftrightarrow}^\tau_{pomhbr} N'$, $(\pi, \sigma) \in Runs(N)$, $(\pi', \sigma') \in Runs(N')$ and $stutt(\pi, \sigma) = R_1 a_1 R_2 \cdots R_{n-1} a_n R_n$, $stutt(\pi', \sigma') = R'_1 a'_1 R'_2 \cdots R'_{m-1} a'_m R'_m$. We say that $stutt(\pi, \sigma)$ and $stutt(\pi', \sigma')$ are *isomorphic*, denoted by $stutt(\pi, \sigma) \simeq stutt(\pi', \sigma')$, if:

1. $n = m$;

2. $\forall i \ (1 \le i \le n) \ a_i = a'_i$;

3. $\forall i \ (1 \le i \le n)$ and $\pi_i \in R_i, \ \pi'_i \in R'_i: \ \pi_i \underline{\leftrightarrow}^\tau_{pomhbr} \pi'_i$.

Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{((\pi, \sigma), (\pi', \sigma')) \mid (\pi, \sigma) \in Runs(N), \ (\pi', \sigma') \in Runs(N'), \ stutt(\pi, \sigma) \simeq stutt(\pi', \sigma')\}$. Let us prove $\mathcal{S} : N \underline{\leftrightarrow}^\tau_{pombpomf} N'$.

1. $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{S}$, since $\pi_N \underline{\leftrightarrow}^\tau_{pomhbr} \pi_{N'}$.

2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{S}$.

   - (back)

     We have $\exists \beta : vis(\rho_C) \simeq vis(\rho_{C'})$. Let $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$. Then $\exists i \ (1 \le i \le n) \ (\tilde{\pi}, \tilde{\sigma}) \in R_i$ from $trace(\pi, \sigma)$. Since $stutt(\pi, \sigma) \simeq stutt(\pi', \sigma')$, then $\exists k \ (1 \le k \le n)$ s.t. R_i corresponds to R'_k from $trace(\pi', \sigma')$. Then $\tilde{\pi} \underline{\leftrightarrow}^\tau_{pomhbr} \pi'(k)$. Consequently, $((\tilde{\pi}, \tilde{\sigma}), (\pi'(k), \sigma'(k))) \in \mathcal{S}$ and $\exists \beta : vis(\rho_{\widetilde{C}}) \simeq vis(\rho_{C'(k)})$. Let us consider the back extension $(\pi'(k), \sigma'(k)) \xrightarrow{\hat{\pi}'} (\pi', \sigma')$. Since β and $\tilde{\beta}$ are isomorphisms, we have $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$.

   - (forth) Obviously.

3. As item 2, but the roles of N and N' are reversed.

($\Rightarrow$) Let $N \underline{\leftrightarrow}^\tau_{pombpomf} N'$. Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{(\pi, \pi', \beta^{\sigma'}_\sigma) \mid (\pi, \sigma) \underline{\leftrightarrow}^\tau_{pombpomf} (\pi', \sigma')\}$. Let us prove $\mathcal{S} : N \underline{\leftrightarrow}^\tau_{pomhbr} N'$.

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{S}$ since $\beta^\varepsilon_\varepsilon = \emptyset$ and $(\pi_N, \varepsilon) \underline{\leftrightarrow}^\tau_{pombpomf} (\pi_{N'}, \varepsilon)$.

2. Let $(\pi, \pi', \beta^{\sigma'}_\sigma) \in \mathcal{S}$. Then by definition of $\mathcal{S}$, $(\pi, \sigma) \underline{\leftrightarrow}^\tau_{pombpomf} (\pi', \sigma')$ and back extension $(\pi_N, \varepsilon) \xrightarrow{\pi} (\pi, \sigma)$ is imitated by $(\bar{\pi}', \varepsilon) \xrightarrow{\bar{\pi}'} (\pi', \sigma')$ for some $\bar{\pi}'$ s.t. $\pi_{N'} \Rightarrow \bar{\pi}'$. If $\pi = (C, \varphi)$ and $\bar{\pi}' = (\overline{C}, \bar{\varphi})$, we have $\beta^{\sigma'}_\sigma : vis(\rho_C) \simeq vis(\rho_{\overline{C}'})$. Since $vis(T'_C) = vis(T_{\overline{C}'})$, where $\pi' = (C', \varphi')$, we have $\beta^{\sigma'}_\sigma : vis(\rho_C) \simeq vis(\rho_{C'})$.

3. Let $(\pi, \pi', \beta^{\sigma'}_\sigma) \in \mathcal{S}$ and $\pi \xrightarrow{v} \tilde{\pi}$. Then by definition of $\mathcal{S}$, $(\pi, \sigma) \underline{\leftrightarrow}^\tau_{pombpomf} (\pi', \sigma')$ and $(\pi, \sigma) \to (\tilde{\pi}, \sigma v)$. The following two cases are possible.

   (a) $l_{\widetilde{C}}(v) \ne \tau$.

      Since $N \underline{\leftrightarrow}^\tau_{pombpomf} N'$, we have $\exists v'_i, w'_j \ (1 \le i \le n, \ 1 \le j \le m), \ v', \pi'_1, \pi'_2$ s.t. $(\pi', \sigma') \xrightarrow{v'_1} \cdots \xrightarrow{v'_n} (\pi'_1, \sigma' v'_1 \cdots v'_n) \xrightarrow{v'} (\pi'_2, \sigma' v'_1 \cdots v'_n v') \xrightarrow{w'_1} \cdots \xrightarrow{w'_m} (\tilde{\pi}', \sigma' v'_1 \cdots v'_n v' w'_1 \cdots w'_m)$, $(\tilde{\pi}, \sigma v) \underline{\leftrightarrow}^\tau_{pombpomf} (\tilde{\pi}', \sigma' v'_1 \cdots v'_n v' w'_1 \cdots w'_m)$ and $l_{\widetilde{C}}(v) = l_{\widetilde{C}'}(v')$, $\forall i, j \ (1 \le i \le n, \ 1 \le j \le m) \ l_{\widetilde{C}'}(v'_i) = l_{\widetilde{C}'}(w'_j) = \tau$. Consequently, $\pi' \xrightarrow{v'_1} \cdots \xrightarrow{v'_n} \pi'_1 \xrightarrow{v'} \pi'_2 \xrightarrow{w'_1} \cdots \xrightarrow{w'_m} \tilde{\pi}'$.

      The back extension $(\pi'_2, \sigma' v'_1 \cdots v'_n v') \to (\tilde{\pi}', \sigma' v'_1 \cdots v'_n v' w'_1 \cdots w'_m)$ is imitated by empty back extension of $(\tilde{\pi}, \sigma v)$. Hence, $(\tilde{\pi}, \sigma v) \underline{\leftrightarrow}^\tau_{pombpomf} (\pi'_2, \sigma' v'_1 \cdots v'_n v')$. Therefore $(\tilde{\pi}, \pi'_2, \beta^{\sigma' v'_1 \cdots v'_n v'}_{\sigma v}) \in \mathcal{S}$.
      Let us consider the back extension $(\pi'_1, \sigma' v'_1 \cdots v'_n) \to (\pi'_2, \sigma' v'_1 \cdots v'_n v')$. It is imitated by some back extension $(\bar{\pi}, \bar{\sigma}) \Rightarrow (\pi, \sigma) \to (\tilde{\pi}, \sigma v)$ s.t. $(\bar{\pi}, \bar{\sigma}) \underline{\leftrightarrow}^\tau_{pombpomf} (\pi'_1, \sigma' v'_1 \cdots v'_n)$. Since $(\pi', \sigma') \Rightarrow (\pi'_1, \sigma' v'_1 \cdots v'_n)$ and $(\pi, \sigma) \underline{\leftrightarrow}^\tau_{pombpomf} (\pi', \sigma')$, by Lemma 4.2 we have $(\pi, \sigma) \underline{\leftrightarrow}^\tau_{pombpomf} (\pi'_1, \sigma' v'_1 \cdots v'_n)$. So, we obtain $(\pi, \pi'_1, \beta^{\sigma' v'_1 \cdots v'_n}_\sigma) \in \mathcal{S}$.
      Hence, we have simulation, since $\pi' \Rightarrow \pi'_1 \xrightarrow{a} \tilde{\pi}'_2$ and $(\pi, \pi'_1, \beta^{\sigma' v'_1 \cdots v'_n}_\sigma) \in \mathcal{S}$, $(\tilde{\pi}, \pi'_2, \beta^{\sigma' v'_1 \cdots v'_n v'}_{\sigma v}) \in \mathcal{S}$.

(b) $l_{\widetilde{C}}(v) = \tau$.

Since $N \underleftrightarrow{\tau}_{pombpomf} N'$, we have $\exists \pi_i'$ ($1 \leq i \leq n$) s.t. $(\pi', \sigma') \Rightarrow (\pi_1', \sigma' v_1) \Rightarrow \cdots \Rightarrow (\pi_n', \sigma' v_1' \cdots v_n') = (\tilde{\pi}', \sigma' v_1' \cdots v_n')$ and $(\tilde{\pi}, \sigma v) \underleftrightarrow{\tau}_{pombpomf} (\tilde{\pi}', \sigma' v_1' \cdots v_n')$.

    i. If $n = 0$, we have proved.

    ii. If $n \geq 1$, and the back extension $(\pi_{n-1}', \sigma' v_1' \cdots v_{n-1}') \Rightarrow (\pi_n', \sigma' v_1' \cdots v_n')$ is simulated by the empty back extension of $(\tilde{\pi}, \sigma v)$ we have proved for $n = 1$, and for $n \geq 2$ we shall continue such a reasoning. Two cases are possible.

    In the first case, we shall obtain $(\tilde{\pi}, \sigma v) \underleftrightarrow{\tau}_{pombpomf} (\pi', \sigma')$ and $(\tilde{\pi}, \pi', \beta_{\sigma v}^{\sigma'}) \in \mathcal{S}$.

    In the second case, we shall obtain $\exists m$ ($1 \leq m \leq n - 1$) s.t. $(\tilde{\pi}, \sigma v) \underleftrightarrow{\tau}_{pombpomf} (\pi_m', \sigma' v_1' \ldots v_m')$ and $(\tilde{\pi}, \pi_m', \beta_{\sigma v}^{\sigma' v_1' \ldots v_m'}) \in \mathcal{S}$.

    The back extension $(\pi_{m-1}', \sigma' v_1' \cdots v_{m-1}') \Rightarrow (\pi_m', \sigma' v_1' \cdots v_m')$ is imitated by some back extension $(\bar{\pi}, \bar{\sigma}) \Rightarrow (\pi, \sigma)$ s.t. $(\bar{\pi}, \bar{\sigma}) \underleftrightarrow{\tau}_{pombpomf} (\pi_{m-1}', \sigma' v_1' \cdots v_{m-1}')$. By Lemma 4.2, we have $(\pi, \sigma) \underleftrightarrow{\tau}_{pombpomf} (\pi_{m-1}', \sigma' v_1' \cdots v_{m-1}')$. So, we obtain $(\pi, \pi_{m-1}', \beta_\sigma^{\sigma' v_1' \cdots v_{m-1}'}) \in \mathcal{S}$.

    Hence, we have simulation, since $\pi' \Rightarrow \pi_{m-1}' \xrightarrow{\tau} \tilde{\pi}_m'$ and $(\pi, \pi_{m-1}', \beta_\sigma^{\sigma' v_1' \cdots v_{m-1}'}) \in \mathcal{S}$, $(\tilde{\pi}, \pi_m', \beta_{\sigma v}^{\sigma' v_1' \cdots v_m'}) \in \mathcal{S}$.

4. As item 3, but the roles of N and N' are reversed. $\qquad\square$

B   Proof of Proposition 6.2

Let $\overline{N} = ref(N, a, D)$, $\overline{N}' = ref(N', a, D)$. Let us note that causal nets of processes of SM-nets are simple chains, i.e. nets s.t. each element has exactly one predecessor (except for the unique input place) and one successor (except for the unique output place).

  CONSTRUCTION (*)

1. Let $\bar{\pi} = (\overline{C}, \bar{\varphi}) \in \Pi(\overline{N})$. Then any element of $\overline{C}$, which is not embedded into $P_N \cup T_N$, has the following properties:

   - has a form $\langle e, f \rangle$ ($e \in P_{C_D} \cup T_{C_D}$, $\pi_D = (C_D, \varphi_D) \in \Pi(D)$ and $f \in T_C$, $\pi = (C, \varphi) \in \Pi(N)$) and is embedded into $\langle x, u \rangle$, $x \in T_D \cup (P_D \setminus \{p_{in}, p_{out}\})$, $u \in l_N^{-1}(a)$;

   - has a unique predecessor $\langle e_{min}, f \rangle$ which is embedded into $\langle t_{min}, u \rangle$, $t_{min} \in p_{in}^\bullet$;

   - belongs to the unique maximal chain ϑ (corresponding to the net C_D) originating from $\langle e_{min}, f \rangle$ where all the elements are embedded into that of type $\langle y, u \rangle$, $y \in T_D \cup (P_D \setminus \{p_{in}, p_{out}\})$, and the only connections of ϑ with the rest of the process are:

     - through the input places of $\langle e_{min}, f \rangle$ (always);
     - (a) through the output places of the maximal transition of the chain $\langle e_{max}, f \rangle$ which is embedded in $\langle t_{max}, u \rangle$, $t_{max} \in {}^\bullet p_{out}$;
       (b) unless the chain stops on a maximal place before.

   Consequently, each such chain ϑ containing in the net $\overline{C}$, may be replaced:

   (a) by transition f which is embedded into u, since they have the same inputs and outputs;

   (b) by transition f which is embedded into u, with new output places corresponding to u, since they have the same outputs, and there is nothing after f (in this case, f is a maximal transition).

   The resulting object will be process $\pi = (C, \varphi) \in \Pi(N)$.

2. Since $N \equiv_\star^\tau N'$, $\star \in \{pw, pom\}$, we can always find $\pi' = (C', \varphi') \in \Pi(N')$ and β s.t.:

   - $\beta^{-1} : vis(\rho_{C'}) \sqsubseteq vis(\rho_C)$, if $\star = pw$;
   - $\beta : vis(\rho_C) \simeq vis(\rho_{C'})$, if $\star = pom$.

   We can suppose that all maximal transitions of C' are visible. Otherwise, $\exists \pi_1' = (C_1', \varphi_1') \in \Pi(N')$ with this property s.t. $\pi_1' \Rightarrow \pi'$. Let us note that in this case $vis(\rho_{C'}) \simeq vis(\rho_{C_1'})$. Then take π_1' instead of π'.

3. For any ϑ, constructed previously, let us replace in C' the transition $\beta(f)$ which is embedded into u', by copy ϑ' of the chain ϑ, where names of elements $\langle e, f \rangle$ are replaced by $\langle e, \beta(f) \rangle$. Two cases are possible:

(a) if the chain is complete, $\beta(f)$ and ϑ' have the same outputs (from u');

(b) if the chain is incomplete, i.e. it terminates by place, we drop all output places of $\beta(f)$.

It is possible, since in this case f is maximal, β does not disregard maximal visible transitions in both the cases $\star \in \{pw, pom\}$, hence $\beta(f)$ is also maximal among *visible* transitions.

In addition, all the maximal transitions of C' are visible, hence no invisible transition can be after $\beta(f)$, and it is maximal among *all* transitions.

In both cases $\beta(f)$ and ϑ' have the same inputs (in u').

It is clear that the constructed object is a process $\bar\pi' = (\overline{C}', \bar\varphi') \in \Pi(\overline{N}')$.

4. Let $g \in vis(T_{\overline{C}})$. Let us define a mapping $\bar\beta$ as follows.

$$\bar\beta(g) = \begin{cases} \beta(g), & g \text{ does not belong to any chain;} \\ \langle e, \beta(f) \rangle, & g = \langle e, f \rangle \text{ belongs to some chain } \vartheta. \end{cases}$$

$\square$

(End of Construction (*))

Let $\bar\pi = (\overline{C}, \bar\varphi) \in \Pi(\overline{N})$. Then $\exists \bar\pi' = (\overline{C}', \bar\varphi') \in \Pi(\overline{N}')$ obtained from $\bar\pi$ by Construction (*).
We have to prove the following statements.

- $\bar\beta^{-1} : vis(\rho_{\overline{C}'}) \sqsubseteq vis(\rho_{\overline{C}})$, if $\star = pw$;

- $\bar\beta : vis(\rho_{\overline{C}}) \simeq vis(\rho_{\overline{C}'})$, if $\star = pom$.

First, let us consider the case $\star = pw$.
Let $g, h \in vis(T_{\overline{C}})$. Five cases are possible:

1. g and h do not belong to any chains;

2. g belongs to the chain ϑ, h does not belong to any chain;

3. g does not belong to any chain, h belongs to the chain ϑ;

4. g and h belong to the same chain ϑ;

5. g belongs to the chain ϑ_1, h belongs to the chain ϑ_2 and $\vartheta_1 \neq \vartheta_2$.

Let us consider the case 5, since the cases 1–4 are simpler. Then $g = \langle e_1, f_1 \rangle$, $h = \langle e_2, f_2 \rangle$, where $e_1 \in vis(T_{C_{D1}})$, $e_2 \in vis(T_{C_{D2}})$ for $\pi_{D1} = (C_{D1}, \varphi_{D1})$, $\pi_{D2} = (C_{D2}, \varphi_{D2}) \in \Pi(D)$, $f_1, f_2 \in vis(T_C)$, f_1 and f_2 are refined in $\overline{C}$ into ϑ_1 and ϑ_2 respectively. We have: $\bar\beta(g) \prec_{\overline{C}'} \bar\beta(h) \Rightarrow \bar\beta(\langle e_1, f_1 \rangle) \prec_{\overline{C}'} \bar\beta(\langle e_2, f_2 \rangle) \Rightarrow$ (by definition of $\bar\beta$) $\langle e_1, \beta(f_1) \rangle \prec_{\overline{C}'} \langle e_2, \beta(f_2) \rangle \Rightarrow$ (since the only connections of chains with the rest of the process are through their minimal and maximal transitions) $\langle e_{max1}, \beta(f_1) \rangle \prec_{\overline{C}'} \langle e_{min2}, \beta(f_2) \rangle \Rightarrow$ (by Construction (*)) $\beta(f_1) \prec_{C'} \beta(f_2) \Rightarrow$ (since $\beta^{-1} : vis(\rho_{C'}) \sqsubseteq vis(\rho_C)$) $f_1 \prec_C f_2 \Rightarrow$ (by Construction (*)) $\langle e_{max1}, f_1 \rangle \prec_{\overline{C}} \langle e_{min2}, f_2 \rangle \Rightarrow \langle e_1, f_1 \rangle \prec_{\overline{C}} \langle e_2, f_2 \rangle \Rightarrow g \prec_{\overline{C}} h$.

The case $\star = pom$ is considered analogously with the exception that all the implications are replaced by symbols "if and only if".

Thus, $\forall \bar\pi \in \Pi(\overline{N}) \ \exists \bar\pi' \in \Pi(\overline{N}')$ s.t. it has desirable properties.
In another direction the proof is symmetrical. $\square$

C   Proof of Proposition 6.3

Let $\overline{N} = ref(N, a, D)$, $\overline{N}' = ref(N', a, D)$ and $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{\star ST} N'$, $\star \in \{i, pw, pom\}$.
    CONSTRUCTION (**)

1. Let $(\bar\pi_E, \bar\pi_P) \in ST - \Pi(\overline{N})$ and $\pi_E, \pi_P \in \Pi(N)$ are constructed from $\bar\pi_E$ and $\bar\pi_P$ respectively by part 1 of Construction (*) from Proposition 6.2.

    Claim C.1 $(\pi_E, \pi_P) \in ST - \Pi(N)$.

    Proof. Let $g, h \in vis(T_{C_E})$ and $g \prec_{C_E} h$. Four cases are possible:

(a) $l_{C_E}(g) \neq a \neq l_{C_E}(h)$;

(b) $l_{C_E}(g) = a \neq l_{C_E}(h)$;

(c) $l_{C_E}(g) \neq a = l_{C_E}(h)$;

(d) $l_{C_E}(g) = a = l_{C_E}(h)$.

Let us consider the case (d), since the cases (a)–(c) are simpler. Then g and h are refined in $\overline{C}_E$ into different chains ϑ_1 and ϑ_2 with elements of the form $\langle e_1, g \rangle$ and $\langle e_2, h \rangle$ respectively, where $e_1 \in vis(T_{C_{D1}})$, $e_2 \in vis(T_{C_{D2}})$ for $\pi_{D1} = (C_{D1}, \varphi_{D1})$, $\pi_{D2} = (C_{D2}, \varphi_{D2}) \in \Pi(D)$. We have: $g \prec_{C_E} h \Rightarrow$ (by Construction (*)) $\langle e_{max1}, g \rangle \prec_{\overline{C}_E} \langle e_{min2}, h \rangle \Rightarrow$ (since $(\bar{\pi}_E, \bar{\pi}_P) \in ST - \Pi(\overline{N})$ and $\langle e_{min2}, h \rangle \in T_{\overline{C}_E}$) $\langle e_{max1}, g \rangle \in T_{\overline{C}_P} \Rightarrow$ (by Construction (*)) $g \in T_{C_P}$.

Let $g \in T_{C_E}$ and $l_{C_E}(g) = \tau \Rightarrow$ (since g is not refined in $\overline{C}_E$) $l_{\overline{C}_E}(g) = \tau \Rightarrow$ (since $(\bar{\pi}_E, \bar{\pi}_P) \in ST - \Pi(\overline{N})$) $g \in T_{\overline{C}_P} \Rightarrow$ (since g is not refined in $\overline{C}_E$) $g \in T_{C_P}$. $\qquad\square$

2. Let us find $(\pi'_E, \pi'_P) \in ST - \Pi(N')$ and β s.t. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$.

3. We obtain $\bar{\pi}'_E$, $\bar{\pi}'_P \in \Pi(\overline{N})$ from π'_E and π'_P respectively by part 3 of Construction (*) from Proposition 6.2.

It is possible to apply this construction, since β does not disregard maximal visible transitions in all the cases $\star \in \{i, pw, pom\}$, and all the maximal transitions of ST-processes are visible.

Claim C.2 $(\bar{\pi}'_E, \bar{\pi}'_P) \in ST - \Pi(\overline{N}')$.

Proof. Let $g', h' \in vis(T_{\overline{C}'_E})$ and $g' \prec_{\overline{C}'_E} h'$. Five cases are possible:

(a) g' and h' do not belong to any chains;

(b) g' belongs to the chain ϑ', h' does not belong to any chain;

(c) g' does not belong to any chain, h' belongs to the chain ϑ';

(d) g' and h' belong to the same chain ϑ';

(e) g' belongs to the chain ϑ'_1, h' belongs to the chain ϑ_2 and $\vartheta_1 \neq \vartheta'_2$.

Let us consider the case (e), since the cases (a)–(d) are simpler. Then $g' = \langle e_1, f'_1 \rangle$, $h' = \langle e_2, f'_2 \rangle$, where $e_1 \in vis(T_{C_{D1}})$, $e_2 \in vis(T_{C_{D2}})$ for $\pi_{D1} = (C_{D1}, \varphi_{D1})$, $\pi_{D2} = (C_{D2}, \varphi_{D2}) \in \Pi(D)$, $f'_1, f'_2 \in vis(T_{C'_E})$, f'_1 and f'_2 are refined in $\overline{C}'_E$ into different chains ϑ'_1 and ϑ'_2 respectively. We have: $g' \prec_{\overline{C}'_E} h' \Rightarrow \langle e_1, f'_1 \rangle \prec_{\overline{C}'_E} \langle e_2, f'_2 \rangle \Rightarrow$ (since the only connections of chains with the rest of the process are through their minimal and maximal transitions) $\langle e_{max1}, f'_1 \rangle \prec_{\overline{C}'_E} \langle e_{min2}, f'_2 \rangle \Rightarrow$ (by Construction (*)) $f'_1 \prec_{C'_E} f'_2 \Rightarrow$ (since $(\pi'_E, \pi'_P) \in ST - \Pi(N')$) $f'_1 \in T_{C'_P} \Rightarrow$ (by Construction (*)) $g' = \langle e_1, f'_1 \rangle \in T_{\overline{C}'_P}$.

Let $g' \in T_{C'_E}$ and $l_{\overline{C}'_E}(g') = \tau \Rightarrow$ (since g' is not refined in $\overline{C}'_E$) $l_{C'_E}(g') = \tau \Rightarrow$ (since $(\pi'_E, \pi'_P) \in ST - \Pi(N')$) $g' \in T_{C'_P} \Rightarrow$ (since g' is not refined in $\overline{C}'_E$) $g' \in T_{\overline{C}'_P}$. $\qquad\square$

4. Let $g \in vis(T_{\overline{C}_E})$. Let us define a mapping $\bar{\beta}$ as follows.

$$\bar{\beta}(g) = \begin{cases} \beta(g), & g \text{ does not belong to any chain;} \\ \langle e, \beta(f) \rangle, & g = \langle e, f \rangle \text{ belongs to some chain } \vartheta. \end{cases}$$

$\qquad\square$

(End of Construction (**))

Let $\mathcal{S}$ consists of elements of the form $((\bar{\pi}_E, \bar{\pi}_P), (\bar{\pi}'_E, \bar{\pi}'_P), \bar{\beta})$ which are obtained by Construction (**) from elements $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$. Let us prove $\mathcal{S} : \overline{N} \underline{\leftrightarrow}_{\star ST} \overline{N}'$.

1. Obviously, $((\pi_{\overline{N}}, \pi_{\overline{N}}), (\pi_{\overline{N}'}, \pi_{\overline{N}'}), \emptyset) \in \mathcal{S}$.

2. Let $((\bar{\pi}_E, \bar{\pi}_P), (\bar{\pi}'_E, \bar{\pi}'_P), \bar{\beta}) \in \mathcal{S}$. Obviously, by Construction (**) we have $\bar{\beta} : vis(\rho_{\overline{C}_E}) \asymp vis(\rho_{\overline{C}'_E})$ and $\bar{\beta}(vis(T_{\overline{C}_P})) = vis(T_{\overline{C}'_P})$, since $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.

3. Let $((\bar{\pi}_E, \bar{\pi}_P), (\bar{\pi}'_E, \bar{\pi}'_P), \bar{\beta}) \in \mathcal{S}$ and $(\bar{\pi}_E, \bar{\pi}_P) \to (\tilde{\bar{\pi}}_E, \tilde{\bar{\pi}}_P)$.

   The element $((\bar{\pi}_E, \bar{\pi}_P), (\bar{\pi}'_E, \bar{\pi}'_P), \bar{\beta})$ is obtained from some element $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$ by Construction (**).

   By part 1 of Construction (**) we obtain $(\tilde{\pi}_E, \tilde{\pi}_P) \in ST - \Pi(N)$ from $(\tilde{\bar{\pi}}_E, \tilde{\bar{\pi}}_P)$.

   Obviously, $(\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P)$.

   Since $\mathcal{R} : N \underleftrightarrow{\tau}_{\star ST} N'$, $\star \in \{i, pw, pom\}$, we have: $\exists \tilde{\beta}, (\tilde{\pi}'_E, \tilde{\pi}'_P)$ s.t.: $(\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P)$, $\tilde{\beta}|_{vis(T_{C_E})} = \beta$ and $((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$.

   By part 3 of Construction (**) we obtain $(\tilde{\bar{\pi}}'_E, \tilde{\bar{\pi}}'_P) \in ST - \Pi(\overline{N})$ from $(\tilde{\pi}'_E, \tilde{\pi}'_P)$.

   It is possible to apply this construction, since $\tilde{\beta}$ does not disregard maximal visible transitions in all the cases $\star \in \{i, pw, pom\}$, and all the maximal transitions of ST-processes are visible.

   By part 4 of Construction (**) we obtain $\tilde{\bar{\beta}}$ from $\tilde{\beta}$.

   Claim C.3 $(\bar{\pi}'_E, \bar{\pi}'_P) \to (\tilde{\bar{\pi}}'_E, \tilde{\bar{\pi}}'_P)$.

   Proof. It is enough to prove that $T_{\overline{C}'_E} \subseteq T_{\tilde{\overline{C}}'_E}$, since the proof of the fact $T_{\overline{C}'_P} \subseteq T_{\tilde{\overline{C}}'_P}$ is analogous.

   Let $g' \in T_{\overline{C}'_E}$. Two cases are possible:

   (a) g' does not belong to any chain;
   (b) g' belongs to some chain ϑ'.

   Let us consider the case (b), since the case (a) is trivial. Then $g' = \langle e, f' \rangle$, where $e \in T_{C_D}$ for $\pi_D = (C_D, \varphi_D) \in \Pi(D)$, $f' \in T_{C'_E}$, f' is refined in $\overline{C}'_E$ into ϑ'. We have: $g' = \langle e, f' \rangle = (\exists f \in T_{C_E}, \beta(f) = f')$ $\langle e, \beta(f) \rangle = (\text{since } (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \text{ implies } f \in T_{\tilde{C}_E}) \langle e, \tilde{\beta}(f) \rangle \in (\text{by definition of } \tilde{\bar{\pi}}'_E) T_{\tilde{\overline{C}}'_E}$. $\square$

   Claim C.4 $\tilde{\bar{\beta}}|_{vis(T_{\overline{C}_E})} = \bar{\beta}$.

   Proof. Let $g \in vis(T_{\overline{C}_E})$. Two cases are possible:

   (a) g does not belong to any chain;
   (b) g belongs to some chain ϑ.

   Let us consider the case (b), since the case (a) is trivial. Then $g = \langle e, f \rangle$, where $e \in vis(T_{C_D})$ for $\pi_D = (C_D, \varphi_D) \in \Pi(D)$, $f \in vis(T_{C_E})$, f is refined in $\overline{C}_E$ into ϑ. We have: $\tilde{\bar{\beta}}(\langle e, f \rangle) = \langle e, \tilde{\beta}(f) \rangle = (\text{since } f \in vis(T_{C_E}) \text{ and } \tilde{\beta}|_{vis(T_{C_E})} = \beta) \langle e, \beta(f) \rangle = (\text{by definition of } \bar{\beta}) \bar{\beta}(\langle e, f \rangle)$. $\square$

   Claim C.5 $((\tilde{\bar{\pi}}_E, \tilde{\bar{\pi}}_P), (\tilde{\bar{\pi}}'_E, \tilde{\bar{\pi}}'_P), \tilde{\bar{\beta}}) \in \mathcal{S}$.

   Proof. Obviously, by Construction (**). $\square$

   Let $\bar{\pi}_P \xrightarrow{\bar{\pi}} \tilde{\bar{\pi}}_E$, $\bar{\pi}'_P \xrightarrow{\bar{\pi}'} \tilde{\bar{\pi}}'_E$, where $\bar{\pi} = (\overline{C}, \bar{\varphi})$, $\bar{\pi}' = (\overline{C}', \bar{\varphi}')$ and $\gamma = \tilde{\bar{\beta}}|_{vis(T_{\overline{C}})}$.
   We have to prove the following statements.

   - $\gamma^{-1} : vis(\rho_{\overline{C}'}) \sqsubseteq vis(\rho_{\overline{C}})$, if $\star = pw$;
   - $\gamma : vis(\rho_{\overline{C}}) \simeq vis(\rho_{\overline{C}'})$, if $\star \in \{i, pom\}$.

   The following two claims are helpful.

   Remark C.1 *Since by Claim C.4 $\tilde{\bar{\beta}}|_{vis(T_{\overline{C}_E})} = \bar{\beta}$ and from $(\bar{\pi}_E, \bar{\pi}_P) \in ST - \Pi(\overline{N})$ follows $\bar{\beta}(vis(T_{\overline{C}_P})) = vis(T_{\overline{C}'_P})$, we have $\tilde{\bar{\beta}}(vis(T_{\tilde{\overline{C}}_E} \setminus T_{\overline{C}_P})) = vis(T_{\tilde{\overline{C}}'_E} \setminus T_{\overline{C}'_P})$.*

   Hence, $\tilde{\bar{\beta}}(vis(T_{\overline{C}})) = vis(T_{\overline{C}'})$.

Remark C.2 *Since $f \in vis(T_{C_P})$ implies $\langle e, f \rangle \in vis(T_{\overline{C}_P})$, then $\langle e, f \rangle \notin vis(T_{\overline{C}_P})$ implies $f \notin vis(T_{C_P})$. Hence, $\langle e, f \rangle \in vis(T_{\widetilde{\overline{C}}_E} \setminus T_{\overline{C}_P} = T_{\overline{C}})$ implies $f \in vis(T_{\widetilde{C}_E} \setminus T_{C_P}) = vis(T_C)$.*

The rest of the proof is analogous to that of from Proposition 6.2.

4. As item 3, but the roles of $\overline{N}$ and $\overline{N}'$ are reversed. $\qquad \square$

D Proof of Proposition 6.7

Let $\overline{N} = ref(N, a, D)$, $\overline{N}' = ref(N', a, D)$. Let us note that occurrence nets of branching processes of SM-net are trees, i.e. nets with exactly one predecessor of each element (with exception of the unique input place).

CONSTRUCTION (***)

1. Let $\bar\varpi = (\overline{O}, \bar\psi) \in \wp_{max}(\overline{N})$. Then each element of $\overline{O}$ which is not embedded into $P_N \cup T_N$, has the following properties:

   - has a form $\langle e, f \rangle$ ($e \in P_{O_D} \cup T_{O_D}$, $\varpi_D = (O_D, \psi_D) \in \wp_{max}(D)$ and $f \in T_O$, $\varpi = (O, \psi) \in \wp_{max}(N)$) and is embedded into $\langle x, u \rangle$, $x \in T_D \cup (P_D \setminus \{p_{in}, p_{out}\})$, $u \in l_N^{-1}(a)$;

   - has a unique predecessor $\langle e_{min}^i, f \rangle$ ($1 \leq i \leq n$) which is embedded into $\langle t_{min}^i, u \rangle$, $t_{min}^i \in p_{in}^\bullet$;

   - belongs to the unique maximal tree ϑ^i (belonging to the set of trees $\vartheta = \cup_{i=1}^n \vartheta^i$, which corresponds to the net O_D), originating from $\langle e_{min}^i, f \rangle$, where all elements are embedded into elements of the form $\langle y, u \rangle$, $y \in T_D \cup (P_D \setminus \{p_{in}, p_{out}\})$ and the only connections of ϑ^i with the rest of the process are:

     – through the input places of $\langle e_{min}^i, f \rangle$ (always);

     – through the output places of maximal elements of tree $\langle e_{max}^{ij}, f \rangle$ ($1 \leq j \leq m$), which are transitions embedded into $\langle t_{max}^{ij}, u \rangle$, $t_{max}^{ij} \in {}^\bullet p_{out}$.

   Let us note that all e_{min}^i ($1 \leq i \leq n$) have the same connections with the rest of the process (as well as all e_{max}^{ij} ($1 \leq i \leq n$, $1 \leq j \leq m$)). Consequently, each such set of trees ϑ containing in $\overline{O}$, may be replaced by transition f which is embedded into u, since they have the same inputs and outputs. The resulting object will be branching process $\varpi = (O, \psi) \in \wp_{max}(N)$.

2. Since $N \equiv_{mes}^\tau N'$, we can always find $\varpi' = (O', \psi') \in \wp_{max}(N')$ and β s.t. $\beta : vis(\xi_O) \simeq vis(\xi_{O'})$.

3. For ϑ, constructed previously, let us replace in O' the transition $\beta(f)$, which is embedded into u', by a copy ϑ' of ϑ, where all names of elements $\langle e, f \rangle$ are replaced by $\langle e, \beta(f) \rangle$. Then $\beta(f)$ and ϑ' have the same outputs (from u') and the same inputs (in u').

   It is clear that the constructed object is branching process $\bar\varpi' = (\overline{O}', \bar\psi') \in \wp_{max}(\overline{N}')$.

4. Let $g \in vis(T_{\overline{O}})$. Let us define a mapping $\bar\beta$ as follows.

$$\bar\beta(g) = \begin{cases} \beta(g), & g \text{ does not belong to any set of trees;} \\ \langle e, \beta(f) \rangle, & g = \langle e, f \rangle \text{ belongs to some set of trees } \vartheta. \end{cases}$$

$\qquad \square$

(End of Construction (***))

Let $\bar\varpi = (\overline{O}, \bar\psi) \in \wp(\overline{N})$. Then $\exists \bar\varpi' = (\overline{O}', \bar\psi') \in \wp(\overline{N}')$ obtained from $\bar\varpi$ by Construction (***).
We have to prove the following statement: $\bar\beta : vis(\xi_{\overline{O}}) \simeq vis(\xi_{\overline{O}'})$.
Let $g, h \in vis(T_{\overline{O}})$. Five cases are possible:

1. g and h do not belong to any sets of trees;

2. g belongs to the set of trees ϑ, h does not belong to any set of trees;

3. g does not belong to any set of trees, h belongs to the set of trees ϑ;

4. g and h belong to the same set f trees ϑ;

5. g belongs to the set of trees ϑ_1, h belongs to the set of trees ϑ_2 and $\vartheta_1 \neq \vartheta_2$.

Let us consider the case 5, since the cases 1–4 are simpler. Then $g = \langle e_1, f_1 \rangle$, $h = \langle e_2, f_2 \rangle$, where $e_1 \in vis(T_{O_{D1}})$, $e_2 \in vis(T_{O_{D2}})$ for $\varpi_{D1} = (O_{D1}, \psi_{D1})$, $\varpi_{D2} = (O_{D2}, \psi_{D2}) \in \wp(D)$, $f_1, f_2 \in vis(T_O)$, f_1 and f_2 are refined in $\overline{O}$ in different sets of trees ϑ_1 and ϑ_2 respectively. Let us prove the preservation of precedence and conflict relations.

- $g \prec_{\overline{O}} h \Leftrightarrow \langle e_1, f_1 \rangle \prec_{\overline{O}} \langle e_2, f_2 \rangle \Leftrightarrow$ (since the only connections of ϑ_1 and ϑ_2 with rest of the process are through their minimal and maximal transitions and all minimal (maximal) transitions have the same connections with the rest) $\forall i, j, k \ \langle e_{max1}^{ij}, f_1 \rangle \prec_{\overline{O}} \langle e_{min2}^{k}, f_2 \rangle \Leftrightarrow$ (by Construction (***)) $f_1 \prec_O f_2 \Leftrightarrow$ (since $\beta : vis(\xi_O) \simeq vis(\xi_{O'})$) $\beta(f_1) \prec_{O'} \beta(f_2) \Leftrightarrow$ (by Construction (***)) $\forall i, j, k \ \langle e_{max1}^{ij}, \beta(f_1) \rangle \prec_{\overline{O}'} \langle e_{min2}^{k}, \beta(f_2) \rangle \Leftrightarrow \langle e_1, \beta(f_1) \rangle \prec_{\overline{O}'} \langle e_2, \beta(f_2) \rangle \Leftrightarrow$ (by definition of $\bar\beta$) $\bar\beta(\langle e_1, f_1 \rangle) \prec_{\overline{O}'} \bar\beta(\langle e_2, f_2 \rangle) \Leftrightarrow \bar\beta(g) \prec_{\overline{O}'} \bar\beta(h)$.

- $g \#_{\overline{O}} h \Leftrightarrow \langle e_1, f_1 \rangle \#_{\overline{O}} \langle e_2, f_2 \rangle \Leftrightarrow$ (since the only connections of ϑ_1 and ϑ_2 with rest of the process are through their minimal and maximal transitions and all minimal (maximal) transitions have the same connections with the rest) $\forall i, k \ \langle e_{min1}^{i}, f_1 \rangle \#_{\overline{O}} \langle e_{min2}^{k}, f_2 \rangle \Leftrightarrow$ (by Construction (***)) $f_1 \#_O f_2 \Leftrightarrow$ (since $\beta : vis(\xi_O) \simeq vis(\xi_{O'})$) $\beta(f_1) \#_{O'} \beta(f_2) \Leftrightarrow$ (by Construction (***)) $\forall i, k \ \langle e_{min1}^{i}, \beta(f_1) \rangle \#_{\overline{O}'} \langle e_{min2}^{k}, \beta(f_2) \rangle \Leftrightarrow \langle e_1, \beta(f_1) \rangle \#_{\overline{O}'} \langle e_2, \beta(f_2) \rangle \Leftrightarrow$ (by definition of $\bar\beta$) $\bar\beta(\langle e_1, f_1 \rangle) \#_{\overline{O}'} \bar\beta(\langle e_2, f_2 \rangle) \Leftrightarrow \bar\beta(g) \#_{\overline{O}'} \bar\beta(h)$.

Thus, $\forall \bar\varpi \in \wp_{max}(\overline{N}) \ \exists \bar\varpi' \in \wp_{max}(\overline{N}')$ s.t. it has desirable properties.
In another direction the proof is symmetrical. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □