

# Discrete time stochastic Petri nets: a model for analysis of stochastic concurrent systems

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**Abstract:** Stochastic Petri nets (SPNs) are an extension of Petri nets (PNs) with an ability of performance (quantitative) analysis.

Behavior analysis: via stochastic process corresponding to an SPN.

Kinds of SPNs:

discrete and continuous timing,

deterministic and stochastic time transition delays,

inhibitor arcs and transition priorities.

Discrete Time SPNs (DTSPNs):

discrete geometric distribution of transition delays,

concurrent transition firing,

the associated processes are discrete time Markov chains (DTMCs).

Application examples and areas are presented.

Notions of labeling and probabilistic equivalences are discussed.

**Keywords:** probability distributions, Markov processes and chains, transient and stationary behaviour, stochastic Petri nets, discrete timing, labeling, probabilistic equivalences.

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## Previous work

Discrete time (subsets of  $\mathbb{N}$ ): step semantics

- *Discrete time stochastic Petri nets (DTSPNs)* [Mol85,ZG94]:  
geometric transition firing delays,  
*Discrete time Markov chain (DTMC)*.
- *Deterministic discrete time stochastic Petri nets (DDTSPNs)* [ZCH97,ZFH01]:  
geometric and deterministic transition firing delays.

## Foundations of probability theory

Probability theory: [Bor86]. Formal methods: [Mar90,Her01,Hav01].

**Definition 1** Probability distribution function (PDSF) of a RV  $\xi$  is:

$$F_{\xi}(x) = P(\xi < x).$$

**Definition 2** Probability mass function (PMF) of a discrete RV:

$$p_{\xi}(x_i) = P(\xi = x_i) \quad (i \in \mathbb{N}).$$

PMF of a discrete RV in vector form:  $p_{\xi} = (p_{\xi}(x_1), p_{\xi}(x_2), \dots)$ .

For discrete RV  $\xi$  PDSF is

$$F_{\xi}(x_n) = \sum_{i=0}^{n-1} p_{\xi}(x_i).$$

**Definition 3** Mean value (MV) of a discrete RV  $\xi$  is

$$M(\xi) = \sum_{i=0}^{\infty} x_i p_{\xi}(x_i),$$

if the series is absolute summarizable.

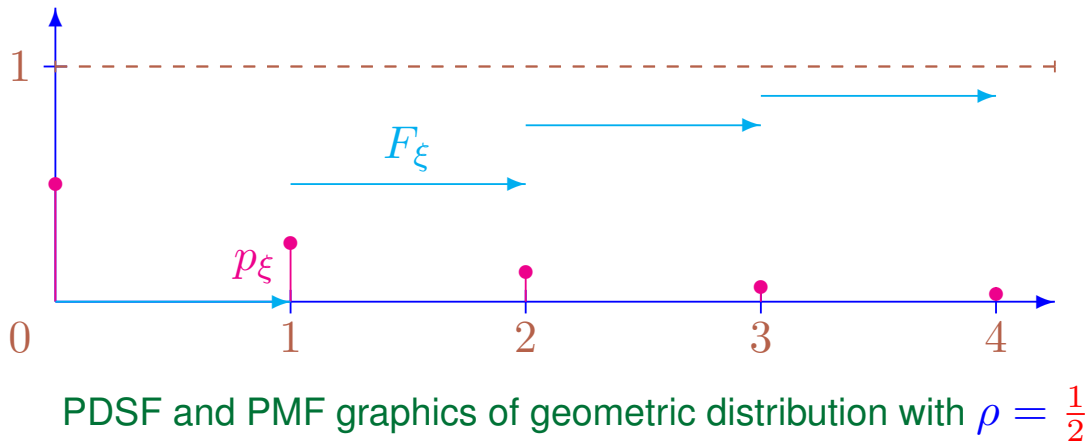
**Definition 4** Variance of RV  $\xi$  is

$$D(\xi) = M((\xi - M(\xi))^2).$$

For discrete RV  $\xi$  its variance is

$$D(\xi) = \sum_{i=0}^{\infty} (x_i - M(\xi))^2 p_{\xi}(x_i).$$

The following holds:  $D(\xi) = M(\xi^2) - (M(\xi))^2$ .



Discrete *geometric distribution*:

$$F_{\xi}(n) = P(\xi < n) = 1 - \rho^n \quad (\rho \in (0; 1), n \in \mathbb{N})$$

$$p_{\xi}(i) = P(\xi = i) = \rho^i(1 - \rho) \quad (i \in \mathbb{N})$$

$$M(\xi) = \sum_{i=0}^{\infty} ip_{\xi}(i) = \frac{\rho}{1 - \rho}$$

$$D(\xi) = \sum_{i=0}^{\infty} (i - M(\xi))^2 p_{\xi}(i) = \frac{\rho}{(1 - \rho)^2}$$

## Stochastic processes

Definitions of Stochastic processes and Markov chains: [Bor86,Mar90,Her01].

**Definition 5** Let  $\Delta$  be a set of parameters (indices) and  $S$  be a set of states. Stochastic process is a set of RVs  $\{\xi(\delta) \mid \delta \in \Delta\} \subseteq S$ .

Usual interpretation:  $\delta$  is time,  $\Delta$  is a time scale (discrete  $\mathbb{N}$  or continuous  $\mathbb{R}_+$ ),  $S$  is a set of all states of RV  $\xi(\delta)$ .

Stochastic processes: discrete or continuous by type of set of states.

Stochastic chain is a stochastic process with discrete set of states.

Stochastic chains: discrete or continuous, depends on time scale.

Stochastic process is stationary, if its properties do not change with simultaneous shift of all states along time scale.

Probabilistic characterization of stochastic processes: hard task.

Special classes of stochastic processes:

- Gauss: multi-factor processes of nature;
- Markov: dynamic of resource sharing systems.



**Definition 6** Let for sets of indices  $\Delta$ , states  $S$  and numbers  $i \in \mathbb{N}$  holds  $\delta_0, \dots, \delta_{i-1}, \delta_i \in \Delta$  ( $\delta_0 < \dots < \delta_{i-1} < \delta_i$ ),  $s_0, \dots, s_{i-1}, s_i \in S$ .

Markov process (MP) is a stochastic process with

Markov property (post-effect absence, memoryless)

$$P(\xi(\delta_i) = s_i \mid \xi(\delta_0) = s_0, \dots, \xi(\delta_{i-1}) = s_{i-1}) =$$

$$P(\xi(\delta_i) = s_i \mid \xi(\delta_{i-1}) = s_{i-1}).$$

Markov chain (MC) is a MP with a discrete set of states.

Discrete time MC (DTMC) is a MC with state changes on finite or countable sets.

MC is (time-)homogeneous, if state change probabilities do not depend on moments when they happen ( $\delta \in \mathbb{N}$  for DTMCs or  $\delta \in \mathbb{R}_+$  for CTMCs):

$$P(\xi(\delta_i) = s_i \mid \xi(\delta_j) = s_j) = P(\xi(\delta_i + \delta) = s_i \mid \xi(\delta_j + \delta) = s_j).$$

Furthermore, all MCs are considered to be homogeneous.

## Discrete time Markov chains (DTMCs)

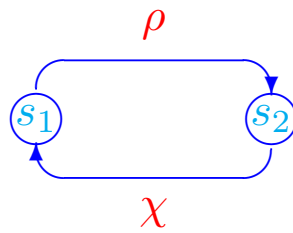
Geometric distribution is the only discrete one with **memoryless property**

$$P(\xi = i + j \mid \xi > j) = P(\xi = i) \quad (i, j \in \mathbb{N}, i \geq 1).$$

Complete probabilistic description of a DTMC: **PMF** over set of states

$S = \{s_1, \dots, s_n\}$  at the **initial time moment** and one-step (along discrete time scale) **transition probabilities**  $\rho_{ij}$  ( $1 \leq i, j \leq n$ ) from  $s_i$  to  $s_j$ .

**(One-step) transition probability diagram (TPD)** of a DTMC is a labeled oriented graph with **vertices** corresponding to states from  $S$ , and **arcs** labeled by one-step transition probabilities  $\rho_{ij}$  ( $1 \leq i, j \leq n$ ). TPD is a **graphical representation** of a DTMC.



### TPD of an example DTMC

**(One-step) transition probability matrix (TPM)** of a DTMC is a matrix  $\mathbf{P}$  of  $n \times n$  over  $[0; 1]$  with one-step transition probabilities

$\rho_{ij} = P(\xi(1) = s_j \mid \xi(0) = s_i)$  ( $1 \leq i, j \leq n$ ) as **elements**.

$$\mathbf{P} = \begin{bmatrix} 1 - \rho & \rho \\ \chi & 1 - \chi \end{bmatrix}.$$

### TPM of an example DTMC

Matrix  $\mathbf{P}^k$  has  $k$ -step transition probabilities as elements

$$\rho_{ij}(k) = \mathbf{P}(\xi(k) = s_j \mid \xi(0) = s_i) \quad (1 \leq i, j \leq n). \quad \mathbf{P}^0 = \mathbf{E}.$$

*Chapman-Kolmogorov equation* establishes a relation between  $k + l$ -step probabilities ( $k, l \in \mathbb{N}$ ) and  $k$ -step and  $l$ -step ones:

$$\mathbf{P}^{k+l} = \mathbf{P}^k \mathbf{P}^l.$$

Probability to stay in  $s_i$  during  $k$  steps and state change at step  $k + 1$  is

$$\rho_{ii}^k (1 - \rho_{ii}).$$

Change a state: success. Stay in a state: failure.

Sojourn time in states of a DTMC is geometrically distributed.

A *DTMC solution*: PMF calculation at **arbitrary time moment** or at **equilibrium conditions**.

Transient behaviour: **transient states**.

Let  $\psi_i(k) = \mathbf{P}(\xi(k) = s_i) (1 \leq i \leq n)$  be probability to come in  $s_i$  during  $k$  steps,  $\psi(k) = (\psi_1(k), \dots, \psi_n(k))$  be its PMF at the moment  $k$ , i.e. its **transient PMF**, and  $\mathbf{P}$  be TPM.

Transient PMF:

$$\psi(k) = \psi(0)\mathbf{P}^k.$$

**Long time system behaviour**: state probabilities could **stabilize (equilibrate)**.

Stationary behaviour: **steady states**.

DTMC is **ergodic**, if steady state PMF exists.

Let  $\psi_i = \lim_{k \rightarrow \infty} \psi_i(k) (1 \leq i \leq n)$  be a probability for an ergodic DTMC to be in steady state  $s_i$ ,  $\psi = (\psi_1, \dots, \psi_n)$  be its **steady state PMF**, and  $\mathbf{P}$  be TPM.

Steady state PMF:

$$\begin{cases} \psi(\mathbf{P} - \mathbf{E}) = 0 \\ \psi \mathbf{e}^T = 1 \end{cases},$$

where  $\mathbf{e}$  is a vector of  $n$  numbers **1**.

## General analysis of DTMCs

1. Find **all states**  $s_i$  ( $1 \leq i \leq n$ ) from  $S$ .
2. Calculate one-step **transition probabilities**  $\rho_{ij}$  from its state  $s_i$  to  $s_j$  ( $1 \leq i, j \leq n$ ).
3. Iteration system of linear equations to analyze its **transient behaviour**.
4. Fixpoint system of linear equations to analyze its **stationary behaviour**.
5. Calculate **state probabilities** analytically or with numerous methods.
6. Calculate standard **performance indices** using state probabilities (throughout, waiting, response time, etc.).

## Formal model

**Definition 7** A discrete time SPN (DTSPN) is a tuple  $N = (P_N, T_N, W_N, \Omega_N, M_N)$ :

- $(P_N, T_N, W_N, M_N)$  is an unlabeled PN;
- $\Omega_N : T_N \rightarrow (0; 1)$  is the transition conditional probability function.

Let  $M$  be a marking of a DTSPN  $N = (P_N, T_N, W_N, \Omega_N, M_N)$ . Then  $t \in \text{Ena}(M)$  fires in the next time moment with probability  $\Omega_N(t)$ , if no other transition is enabled in  $M$ : conditional probability.

Conditional probability to fire in a marking  $M$  for a transition set (not a multiset)  $U \subseteq \text{Ena}(M)$  s.t.  $\bullet U \subseteq M$ :

$$PF(U, M) = \prod_{t \in U} \Omega_N(t) \cdot \prod_{t \in \text{Ena}(M) \setminus U} (1 - \Omega_N(t)).$$

Concurrent transition firings at discrete time moments.

DTSPNs have step semantics.

For all DTSPN  $N = (P_N, T_N, W_N, \Omega_N, M_N)$  we have

$RS(N) = RS(P_N, T_N, W_N, M_N)$ : reachability sets of a DTSPN and its underlying PN coincide.

Qualitative properties of a DTSPNs: analysis of reachability graphs for underlying PNs.

Quantitative properties of a DTSPNs: analysis of DTMCs for *bounded* and *live* DTSPNs.

DTMC  $DTMC(N)$  *corresponding* to a DTSPN  $N$ :

1. Set of states  $S = RS(N)$ .
2. Probability  $\rho_{ij}$  ( $1 \leq i, j \leq n = |S|$ ) of state change from  $M_i$  to  $M_j$  is

$$\rho_{ij} = \frac{\sum_{\{U \subseteq \text{Ena}(M_i) \mid M_i \xrightarrow{U} M_j\}} PF(U, M_i)}{\sum_{\{V \subseteq \text{Ena}(M_i) \mid \bullet V \subseteq M_i\}} PF(V, M_i)}.$$

(One-step) TPM  $\mathbf{P}$  for  $DTMC(N)$  with elements  $\rho_{ij}$ .

Transient ( $k$ -step) PMF for  $DTMC(N)$ :

$$\psi(k) = \psi(0)\mathbf{P}^k,$$

where  $k \in \mathbb{N}$  and  $\psi(0) = (\psi_1(0), \dots, \psi_n(0))$  is a **probability of the initial distribution**,  $\psi_i(0)$  ( $1 \leq i \leq n$ ):

$$\psi_i(0) = \begin{cases} 1 & M_i = M_N \\ 0 & \text{otherwise} \end{cases}.$$

Here  $\psi(k) = (\psi_1(k), \dots, \psi_n(k))$  is a **transient PMF** over  $k$ -step reachable markings, and  $\psi_i(k)$  ( $1 \leq i \leq n$ ) are **transient probabilities** of  $M_i$ .

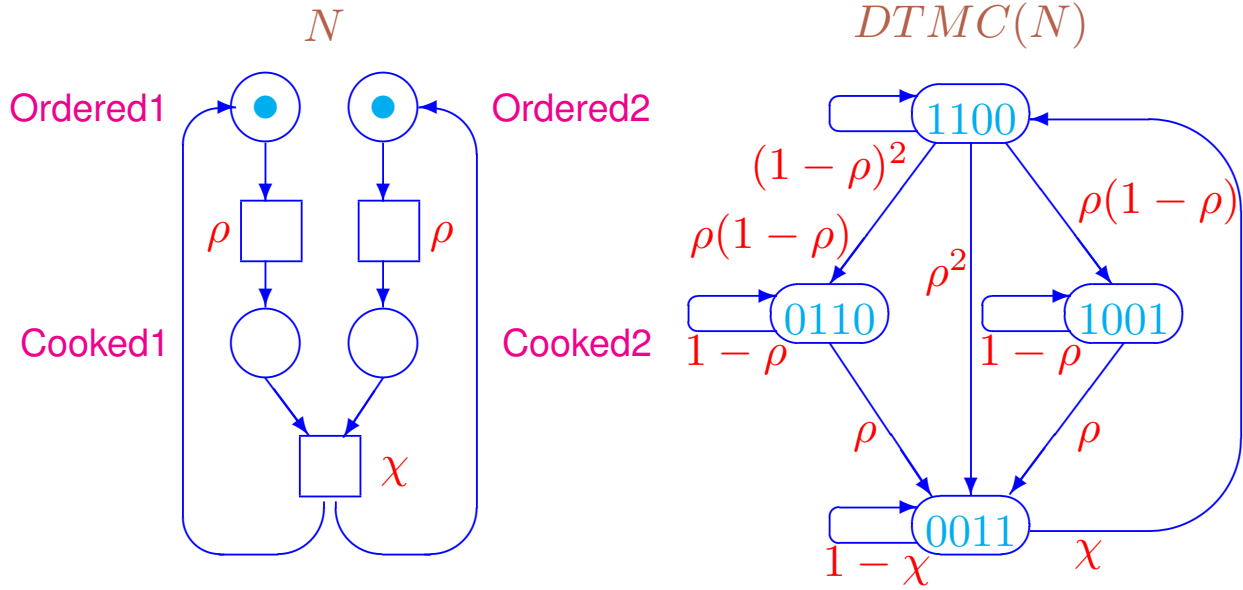
Steady state PMF for  $DTMC(N)$ :

$$\begin{cases} \psi(\mathbf{P} - \mathbf{E}) = 0 \\ \psi \mathbf{e}^T = 1 \end{cases}.$$

Here  $\psi = (\psi_1, \dots, \psi_n)$  is a **steady state PMF** over reachable markings, and  $\psi_i$  ( $1 \leq i \leq n$ ) are **steady state probabilities** of  $M_i$ .



## Illustrative example



DTSPN of restaurant and its DTMC

Restaurant with two-course dinner: DTSPN  $N$ .

First, the dinner is **ordered**.

When **both** dishes have been **cooked**, they are served.

Cooking processes of the dishes are **independent**.

Cooking time is about **equal**.

Places:  $P_N = \{p_1, p_2, p_3, p_4\}$ .

Transitions:  $T_N = \{t_1, t_2, t_3\}$ .

Conditional probabilities:  $\Omega_N(t_1) = \Omega_N(t_2) = \rho$ ,  $\Omega_N(t_3) = \chi$ .

Interpretation of places.

$p_1$ : first dish has been ordered (**Ordered1**).

$p_2$ : second dish has been ordered (**Ordered2**).

$p_3$ : first dish has been cooked (**Cooked1**).

$p_4$ : second dish has been cooked (**Cooked2**).

Interpretation of markings.

$M_1 = (1, 1, 0, 0)$ : both dishes have been ordered (**Ordered**).

$M_2 = (0, 1, 1, 0)$ : first dish has been cooked (**Cooked1**).

$M_3 = (1, 0, 0, 1)$ : second dish has been cooked (**Cooked2**).

$M_4 = (0, 0, 1, 1)$ : both dishes have been cooked (**Cooked**).

Interpretation of transitions and their conditional probabilities.

1. When both dishes have been ordered, **first dish is cooked**:  
 $t_1$  with probability  $\rho$ .
2. When both dishes have been ordered, **second dish is cooked**:  
 $t_2$  with probability  $\rho$ .
3. When both dishes have been cooked, **they are served**:  
 $t_3$  with probability  $\chi$ .

One-step TPM for DTMC  $DTMC(N)$  is

$$\mathbf{P} = \begin{bmatrix} (1 - \rho)^2 & \rho(1 - \rho) & \rho(1 - \rho) & \rho^2 \\ 0 & 1 - \rho & 0 & \rho \\ 0 & 0 & 1 - \rho & \rho \\ \chi & 0 & 0 & 1 - \chi \end{bmatrix}$$

Steady-state PMF for DTMC  $DTMC(N)$  is the solution of equation system

$$\left\{ \begin{array}{l} \rho(2 - \rho)\psi_1 = \chi\psi_4 \\ \rho(1 - \rho)\psi_1 = \rho\psi_2 \\ \rho(1 - \rho)\psi_1 = \rho\psi_3 \\ \rho^2\psi_1 + \rho\psi_2 + \rho\psi_3 = \chi\psi_4 \\ \psi_1 + \psi_2 + \psi_3 + \psi_4 = 1 \end{array} \right.$$

The result is

$$\psi = \frac{1}{\chi(3 - 2\rho) + \rho(2 - \rho)}(\chi, \chi(1 - \rho), \chi(1 - \rho), \rho(2 - \rho)).$$

The case  $\rho = \chi$ :

$$\psi = \frac{1}{7}(2, 1, 1, 3).$$

## Transition labeling

DTSPNs are standardly **unlabeled**:

**acceptable** to model **logically different** activities:

transitions  $t_1$  and  $t_3$  of DTSPN from restaurant example;

**not acceptable** to model **logically equal** activities:

transitions  $t_1$  and  $t_2$  of DTSPN from restaurant example.

Transition labeling:

$L_N(t_1) = L_N(t_2) = \textit{Cook}$ ,  $L_N(t_3) = \textit{Serve}$ .

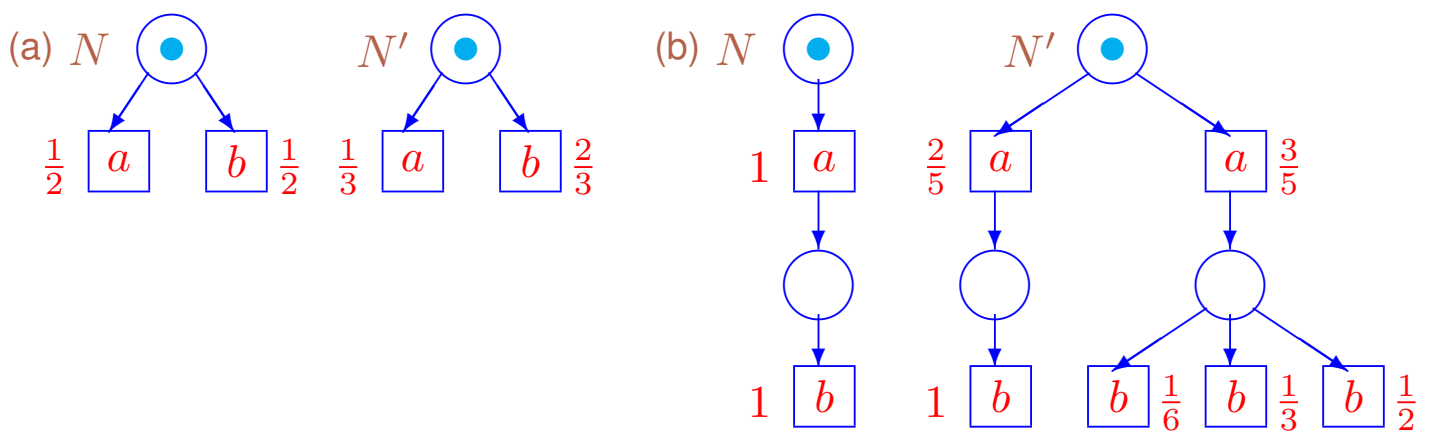
Conditional probabilities are associated with **actions**:

*Cook* has probability  $\rho$ , and *Serve* has  $\chi$ .

Transition concurrency in DTSPNs: **step semantics** for **labeled** DTSPNs.

Definition of DTSPN transition labeling: [BT01].

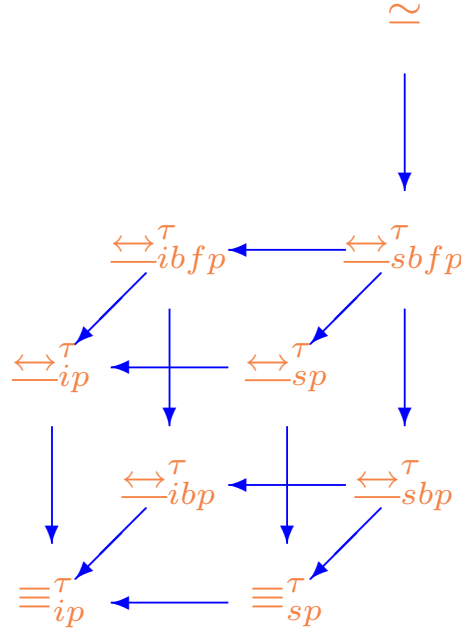
## Properties of probabilistic relations



## PP: Properties of probabilistic equivalences

- In Figure PP(a) LDTSPNs  $N$  and  $N'$  could not be related by any (even trace) probabilistic equivalence, since only in  $N'$  action  $a$  has probability  $\frac{1}{3}$ .
- In Figure PP(b) LDTSPNs  $N$  and  $N'$  are related by any (even bisimulation) probabilistic equivalence, since in our model probabilities of consequent actions are multiplied, and that of alternative ones are summarized.

## Comparing the probabilistic $\tau$ -equivalences



### Interrelations of the probabilistic $\tau$ -equivalences

**Proposition 1** Let  $\star \in \{i, s\}$ . For LDTSPNs  $N$  and  $N'$

1.  $N \leftrightarrow_{\star p}^\tau N' \Rightarrow N \equiv_{\star p}^\tau N'$ ;
2.  $N \leftrightarrow_{\star bp}^\tau N' \Rightarrow N \equiv_{\star p}^\tau N'$ ;
3.  $N \leftrightarrow_{\star bfp}^\tau N' \Rightarrow N \leftrightarrow_{\star p}^\tau N'$  and  $N \leftrightarrow_{\star bp}^\tau N'$ .

**Theorem 1** Let  $\leftrightarrow, \Leftarrow \in \{\equiv^\tau, \leftrightarrow^\tau, \approx\}$  and

$\star, \star\star \in \{-, ip, sp, ibp, sbp, ibfp, sbfp\}$ . For LDTSPNs  $N$  and  $N'$

$$N \leftrightarrow_\star N' \Rightarrow N \Leftarrow_{\star\star} N'$$

iff in the graph in figure above there exists a directed path from  $\leftrightarrow_\star$  to  $\Leftarrow_{\star\star}$ .

## The results reported

Description of DTSPNs.

Analysis methods and illustrative examples.

Transition labeling

Probabilistic equivalences.

The most perspective model: DTSPNs and their extensions,  
like DDTSPNs [ZCH97,ZFH01].

## Advantages and disadvantages of SPNs

### Advantages

- Convenient for **theoretical reasoning** on behaviour of systems with shared resources and for use in **development tools**.
- Performance can be evaluated from **SPN structure**, and detailed **analysis is accomplished using MC** with well-known algorithms.
- **Applicable when synchronization is important**: analysis of **systems with interacting components**.

### Disadvantages

- **High complexity of large system specification** because of **absence of modularity and intricateness** of the corresponding SPNs.
- **More abstract SPNs** with better expressive power: **analytical and structural restrictions** or **partitioning, simulation and numerical methods**.
- **Concurrency of the PN underlining an SPN is reflected only partially** in the corresponding MC: in the best case, it has **step semantics that is not “true concurrent”**.



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