Equivalences for net models of concurrent stochastic systems

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Abstract: Labeled discrete time stochastic Petri nets (LDTSPNs) are proposed.

The visible behavior of LDTSPNs is described by transition labels.

Trace and bisimulation probabilistic equivalences are introduced.

A diagram of their interrelations is presented.

Some of the equivalences are characterized via formulas of probabilistic modal logics.

The equivalences are used to compare stationary behavior of nets.

Keywords: stochastic Petri nets, step semantics, probabilistic equivalences, bisimulation, modal logics, stationary behavior.

Contents

Previous work

Transition labeling

- CTSPNs [Buc95]
- GSPNs [Buc98]
- DTSPNs [BT00]

Equivalences

- Stochastic automata (SAs) [Buc99]
- Probabilistic transition systems (PTSs) [BM89,Chr90,LS91,BH97,KN98]
- CTMCs [HR94,Hil94]
- CTSPNs [Buc95]
- GSPNs [Buc98]
- Markov process algebras (MPAs) [Buc94]
- Stochastic event structures (SESs) [MCW03]

Probabilistic modal logics

- Logic PML [LS91]

Formal model

Definition 1 *A* Labeled discrete time stochastic Petri net (LDTSPN) *is a tuple* $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$*:*

- P_N *and* T_N *are finite sets of* places *and* transitions *(*$P_N \cup T_N \neq \emptyset,\ P_N \cap T_N = \emptyset$*);*
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to I\!N$ *is the* arc weight *function;*
- $\Omega_N : T_N \to (0; 1]$ *is the* transition probability *function;*
- $L_N : T_N \to Act_\tau$ *is the* transition labeling *function (*$Act_\tau = Act \cup \{\tau\}$*);*
- $M_N \in I\!N_f^{P_N}$ *is the* initial marking*.*

Let M be a marking of a LDTSPN $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$.

Then $t \in Ena(M)$ fires in the next time moment with probability $\Omega_N(t)$, if no other transition is enabled in M: conditional probability.

Conditional probability to fire in a marking M for a transition set (not a multiset) $U \subseteq Ena(M)$ s.t. ${}^{\bullet}U \subseteq M$:

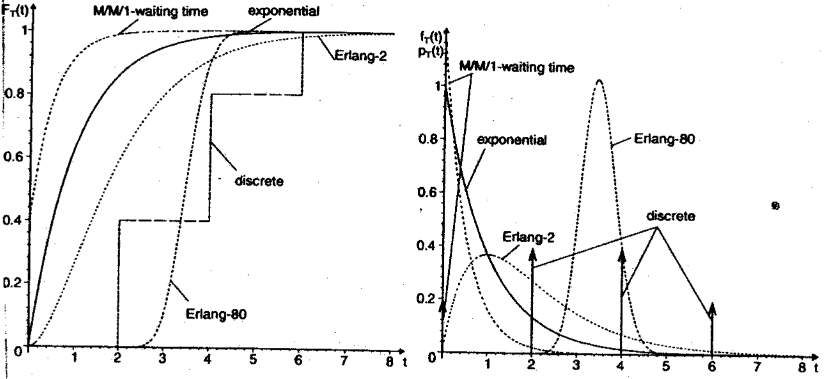
$$PF(U, M) = \prod_{t \in U} \Omega_N(t) \cdot \prod_{t \in Ena(M) \setminus U} (1 - \Omega_N(t)).$$

Concurrent transition firings at discrete time moments.

LDTSPNs have *step* semantics.

Table 3. Some Examples for Random Variables (RV)

Continuous RV	Discrete RV
- Interarrival times of jobs	- Number of buffered jobs
- Activity times	- Idle/busy/overflow-states
- Waiting times	- Arrivals in a fixed interval



ig. 6. Typical Examples for Distribution Functions $F_T(t)$ and their Related ensity Functions $f_T(t)$ or Discrete State Probabilities $p_T(t)$: Exponential d.f., rlangian d.f. of low or high order, discrete time d.f. and a typical d.f. for waiting mes (here from a M/M/1-queuing station).

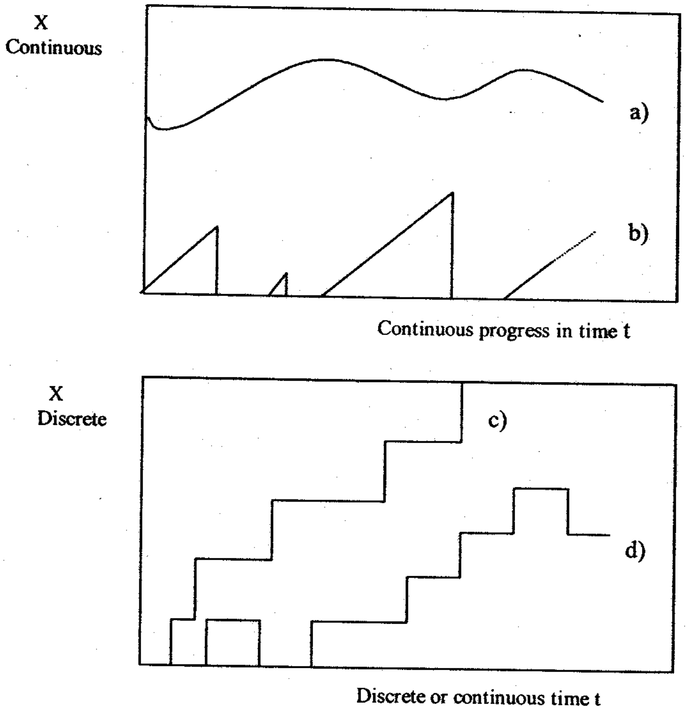


Fig. 7. Stochastic Process Examples: Mean packet delay in the Internet (a), duration of a telephone call (b), counting process (c), number of busy channels in an ATM-network (d).

Example 1. Graphical representation of a DTMC. In Figure 4 we show the state transition diagram for the DTMC with state-transition probability matrix

$$\mathbf{P} = \frac{1}{10}\begin{pmatrix} 6 & 2 & 2 \\ 1 & 8 & 1 \\ 6 & 0 & 4 \end{pmatrix}. \tag{2}$$

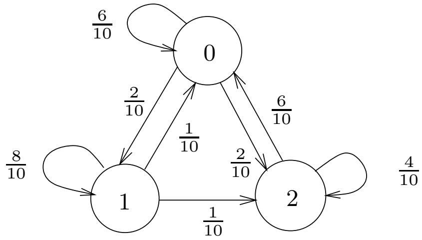
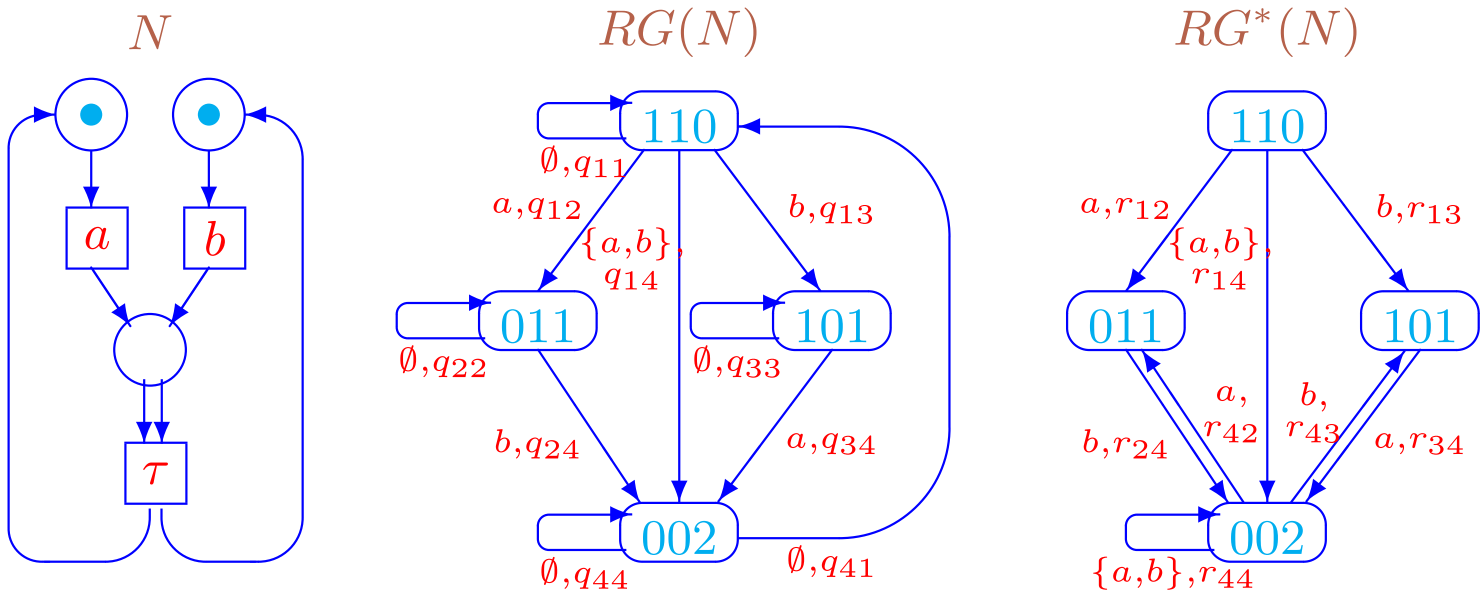


Fig. 4. State transition diagram for the example DTMC (B.R. Haverkort, *Performance of Computer Communication Systems*, 1998. © John Wiley & Sons Limited. Reproduced with Permission.)

Example of LDTSPNs



A LDTSPN and the corresponding reachability graphs

$q_{11} = \overline{\Omega}_N(t_1) \cdot \overline{\Omega}_N(t_2)$ $\quad q_{12} = \Omega_N(t_1) \cdot \overline{\Omega}_N(t_2)$ $\quad q_{13} = \overline{\Omega}_N(t_1) \cdot \Omega_N(t_2)$

$q_{14} = \Omega_N(t_1) \cdot \Omega_N(t_2)$ $\quad q_{22} = \overline{\Omega}_N(t_2)$ $\quad q_{24} = \Omega_N(t_2)$

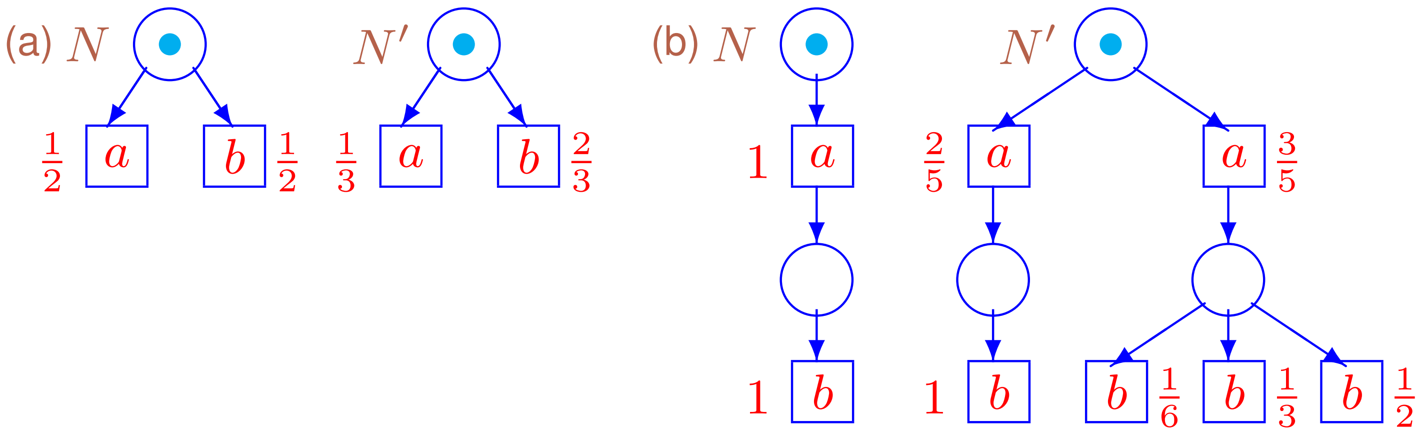
$q_{33} = \overline{\Omega}_N(t_1)$ $\quad q_{34} = \Omega_N(t_1)$ $\quad q_{41} = \Omega_N(t_3)$

$q_{44} = \overline{\Omega}_N(t_3)$

$r_{12} = r_{42} = \frac{q_{12}}{1-q_{11}}$ $\quad r_{13} = r_{43} = \frac{q_{13}}{1-q_{11}}$ $\quad r_{14} = r_{44} = \frac{q_{14}}{1-q_{11}}$

$r_{24} = 1$ $\quad r_{34} = 1$

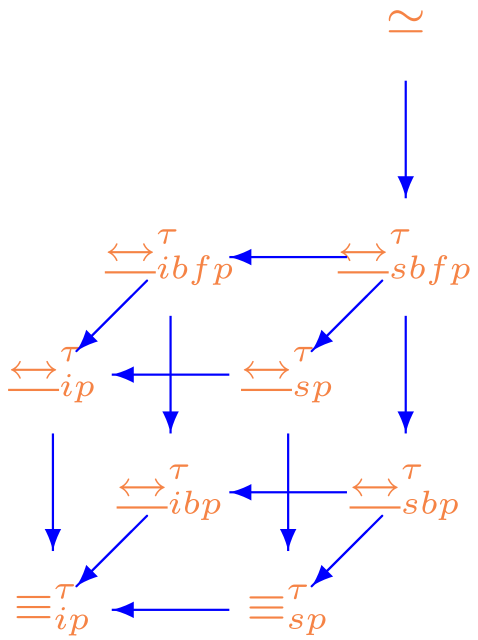
Properties of probabilistic relations



PP: Properties of probabilistic equivalences

- In Figure PP(a) LDTSPNs N and N' could not be related by any (even trace) probabilistic equivalence, since only in N' action a has probability $\frac{1}{3}$.
- In Figure PP(b) LDTSPNs N and N' are related by any (even bisimulation) probabilistic equivalence, since in our model probabilities of consequent actions are multiplied, and that of alternative ones are summarized.

Comparing the probabilistic τ-equivalences



Interrelations of the probabilistic τ-equivalences

Proposition 1 *Let* $\star \in \{i, s\}$*. For LDTSPNs* N *and* N'

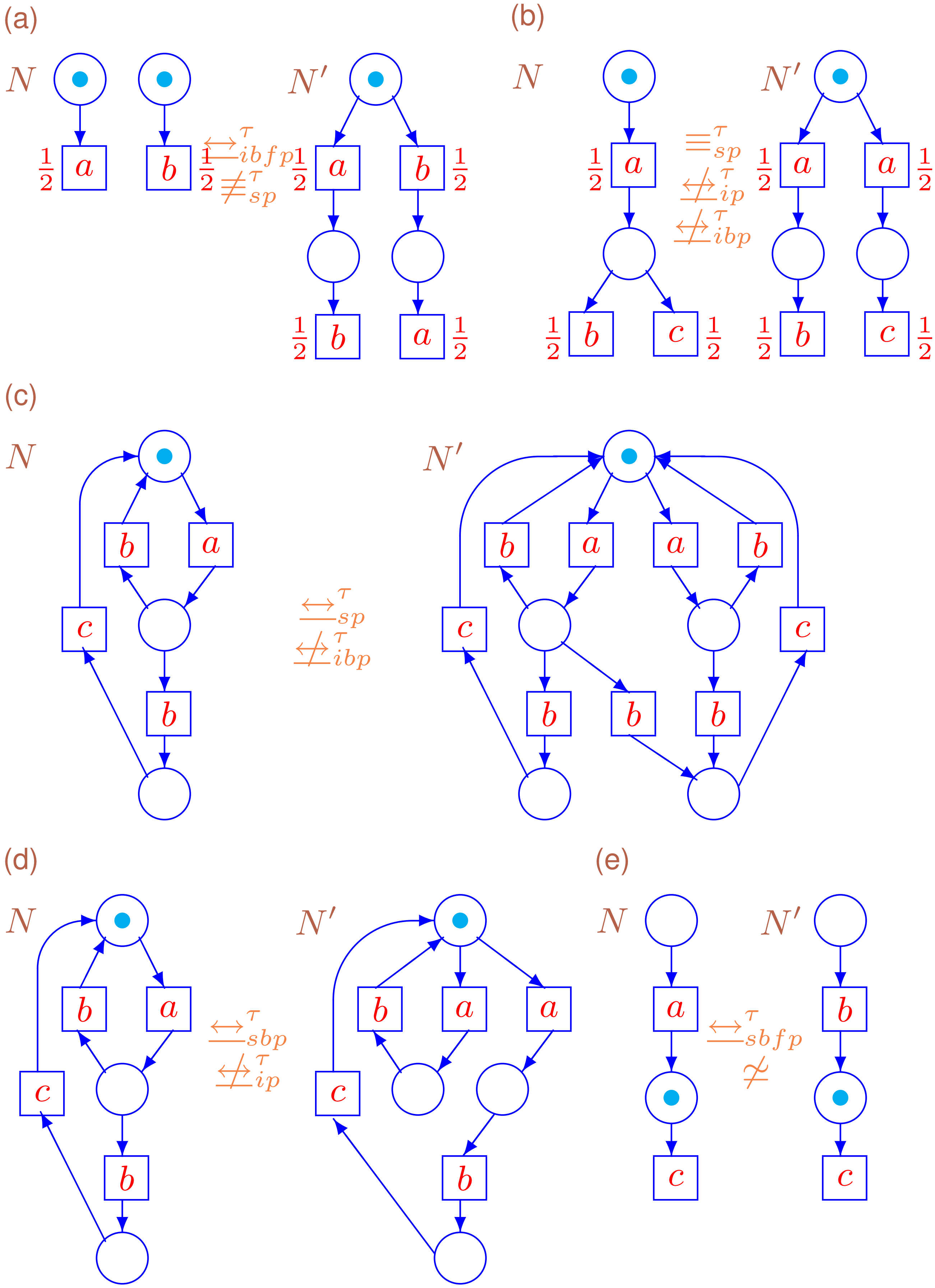
1. $N \underline{\leftrightarrow}^{\tau}_{\star p} N' \;\Rightarrow\; N \equiv^{\tau}_{\star p} N'$;
2. $N \underline{\leftrightarrow}^{\tau}_{\star bp} N' \;\Rightarrow\; N \equiv^{\tau}_{\star p} N'$;
3. $N \underline{\leftrightarrow}^{\tau}_{\star bfp} N' \;\Rightarrow\; N \underline{\leftrightarrow}^{\tau}_{\star p} N'$ and $N \underline{\leftrightarrow}^{\tau}_{\star bp} N'$.

Theorem 1 *Let* $\leftrightarrow, \leftrightarrow\!\!\!\!\rightarrow \in \{\equiv^{\tau}, \underline{\leftrightarrow}^{\tau}, \simeq\}$ *and* $\star, \star\star \in \{\_, ip, sp, ibp, sbp, ibfp, sbfp\}$*. For LDTSPNs* N *and* N'

$$N \leftrightarrow_{\star} N' \;\Rightarrow\; N \leftrightarrow\!\!\!\!\rightarrow_{\star\star} N'$$

iff in the graph in figure above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\leftrightarrow\!\!\!\!\rightarrow_{\star\star}$*.*

Examples of the probabilistic relations



S: Examples of the probabilistic τ-equivalences

- In Figure S(a), $N \underline{\leftrightarrow}^{\tau}_{ibfp} N'$, but $N \not\equiv^{\tau}_{sp} N'$, since only in the LDTSPN N' actions a and b cannot occur concurrently.
- In Figure S(b), $N \equiv^{\tau}_{sp} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{ip} N'$ and $N \underline{\not\leftrightarrow}^{\tau}_{ibp} N'$, since only in the LDTSPN N' an action a can occur so that no action b can occur afterwards.
- In Figure S(c), $N \underline{\leftrightarrow}^{\tau}_{sp} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{ibp} N'$, since only in N' there is a place with two input transitions labeled by b. Hence, the probability for a token to go to this place is always more than for that with only one input b-labeled transition.
- In Figure S(d), $N \underline{\leftrightarrow}^{\tau}_{sbp} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{ip} N'$, since only in the LDTSPN N' an action a can occur so that a sequence of actions bc cannot occur just after it.
- In Figure S(e), $N \underline{\leftrightarrow}^{\tau}_{sbfp} N'$ but $N \not\simeq N'$, since upper transitions of LDTSPNs N and N' are labeled by different actions (a and b).

Logic $IPML$

Definition 2 *$\top$ denotes the truth, $a \in Act$, $\mathcal{P} \in (0;1]$.*

A formula of $IPML$:

$$\Phi ::= \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \langle a\rangle_{\mathcal{P}}\Phi$$

IPML is the set of *all formulas* of $IPML$.

Definition 3 *Let N be a LDTSPN and $M \in RS^*(N)$. The satisfaction relation $\models_N \subseteq RS^*(N) \times \mathbf{IPML}$:*

1. *$M \models_N \top$ — always;*
2. *$M \models_N \neg\Phi$, if $M \not\models_N \Phi$;*
3. *$M \models_N \Phi\wedge\Psi$, if $M \models_N \Phi$ and $M \models_N \Psi$;*
4. *$M \models_N \langle a\rangle_{\mathcal{P}}\Phi$, if $\exists\mathcal{L} \subseteq RS^*(N)$ $M \stackrel{a}{\rightsquigarrow}_{\mathcal{Q}} \mathcal{L}$, $\mathcal{Q} \geq \mathcal{P}$ and $\forall\widetilde{M} \in \mathcal{L}$ $\widetilde{M} \models_N \Phi$.*

$\langle a\rangle\Phi = \exists\mathcal{P} > 0\ \langle a\rangle_{\mathcal{P}}\Phi$.

$\langle a\rangle_{\mathcal{Q}}\Phi$ implies $\langle a\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

We write $N \models_N \Phi$, if $M_N \models_N \Phi$.

Definition **4** *N and N' are* logical equivalent *in $IPML$, $N =_{IPML} N'$, if $\forall \Phi \in \mathbf{IPML}\ N \models_N \Phi \Leftrightarrow N' \models_{N'} \Phi$.*

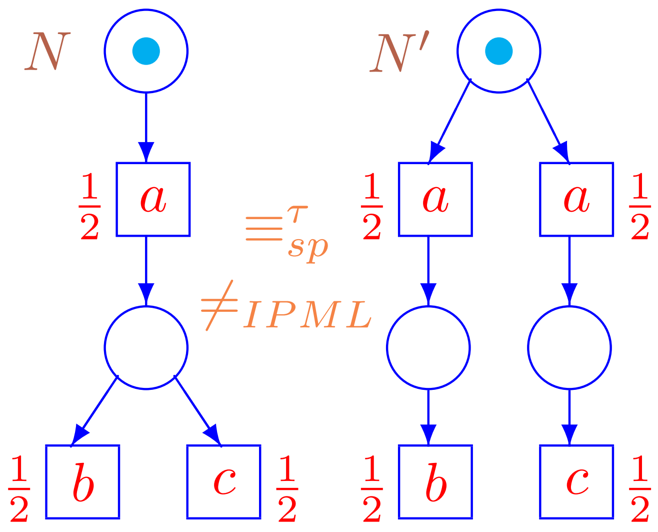
Let for a LDTSPN N $M \in RS^*(N),\ a \in Act$.

The set of *next* to M markings *after occurrence of visible action* a (*visible image set*) is $VisImage(M, a) = \{\widetilde{M} \mid M \stackrel{a}{\twoheadrightarrow} \widetilde{M}\}$.

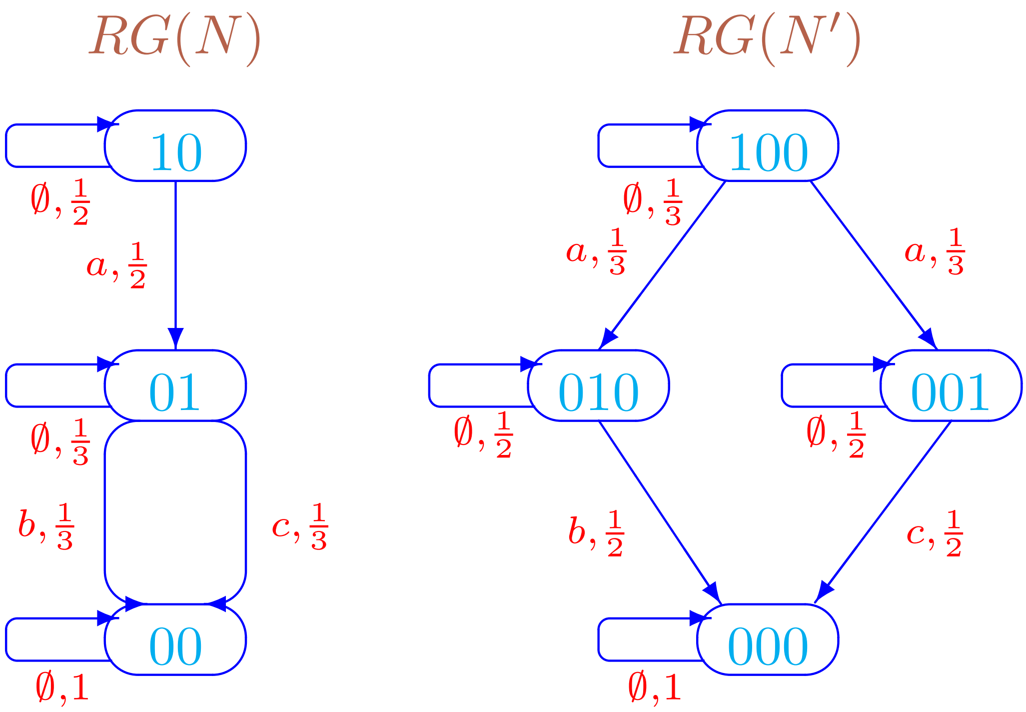
A LDTSPN N is a *image-finite* one, if $\forall M \in RS^*(N)\ \forall a \in Act\ |VisImage(M, a)| < \infty$.

Theorem **2** *For image-finite LDTSPNs N and N'*

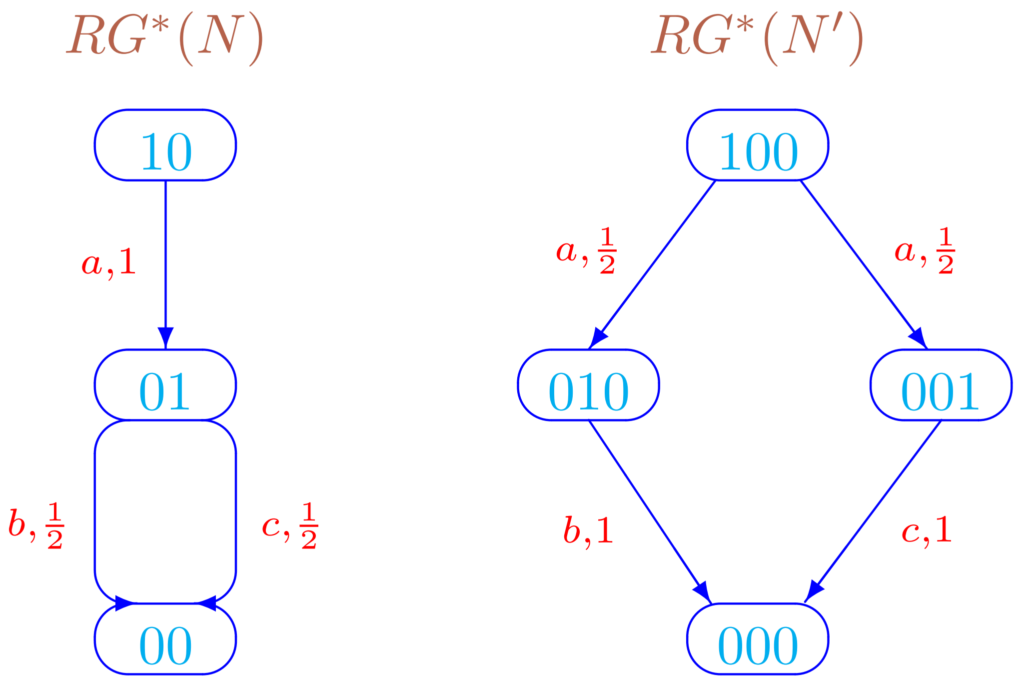
$$N \underline{\leftrightarrow}_{ip}^{\tau} N' \Leftrightarrow N =_{IPML} N'.$$



Differentiating power of $=_{IPML}$



Reachability graphs of the LDTSPNs above



Visible reachability graphs of the LDTSPNs above

$N \equiv^{\tau}_{sp} N'$, but $N \not\equiv_{IPML} N'$, because for $\Phi = \langle a\rangle_1 \langle b\rangle_{\frac{1}{2}} \top,\ N \models_N \Phi$, but $N' \not\models_{N'} \Phi$, since only in N' an action a can occur so that no action b can occur afterwards.

Logic $SPML$

Definition 5 *$\top$ denotes the truth, $A \in I\!N_f^{Act}$, $\mathcal{P} \in (0;1]$.*

A formula of $SPML$:

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SPML is the set of *all formulas* of $SPML$.

Definition 6 *Let N be a LDTSPN and $M \in RS^*(N)$. The satisfaction relation $\models_N \subseteq RS^*(N) \times \mathbf{SPML}$:*

1. $M \models_N \top$ *— always;*
2. $M \models_N \neg\Phi$*, if* $M \not\models_N \Phi$*;*
3. $M \models_N \Phi\wedge\Psi$*, if* $M \models_N \Phi$ *and* $M \models_N \Psi$*;*
4. $M \models_N \langle A\rangle_{\mathcal{P}}\Phi$*, if* $\exists\mathcal{L} \subseteq RS^*(N)\ M \xrightarrow{A}_{\mathcal{Q}} \mathcal{L},\ \mathcal{Q} \geq \mathcal{P}$ *and* $\forall\widetilde{M} \in \mathcal{L}\ \widetilde{M} \models_N \Phi$.

$\langle A\rangle\Phi = \exists\mathcal{P} > 0\ \langle A\rangle_{\mathcal{P}}\Phi$.

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We write $N \models_N \Phi$, if $M_N \models_N \Phi$.

Definition 7 *N and N' are logical equivalent in $SPML$, $N =_{SPML} N'$, if $\forall \Phi \in \mathbf{SPML}\ N \models_N \Phi \ \Leftrightarrow\ N' \models_{N'} \Phi$.*

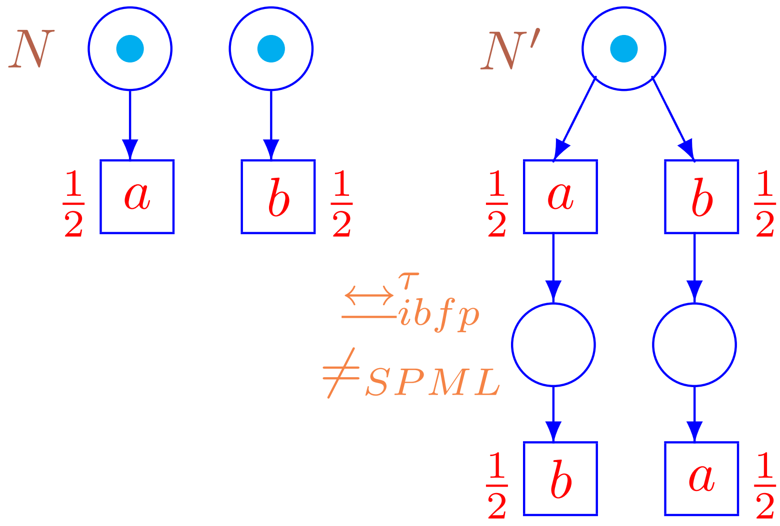
Let for a LDTSPN N $M \in RS^*(N),\ A \in I\!N_f^{Act}$.

The set of *next* to M markings *after occurrence of multiset of visible actions* A (*visible image set*) is
$VisImage(M, A) = \{\widetilde{M} \mid M \stackrel{A}{\twoheadrightarrow} \widetilde{M}\}$.

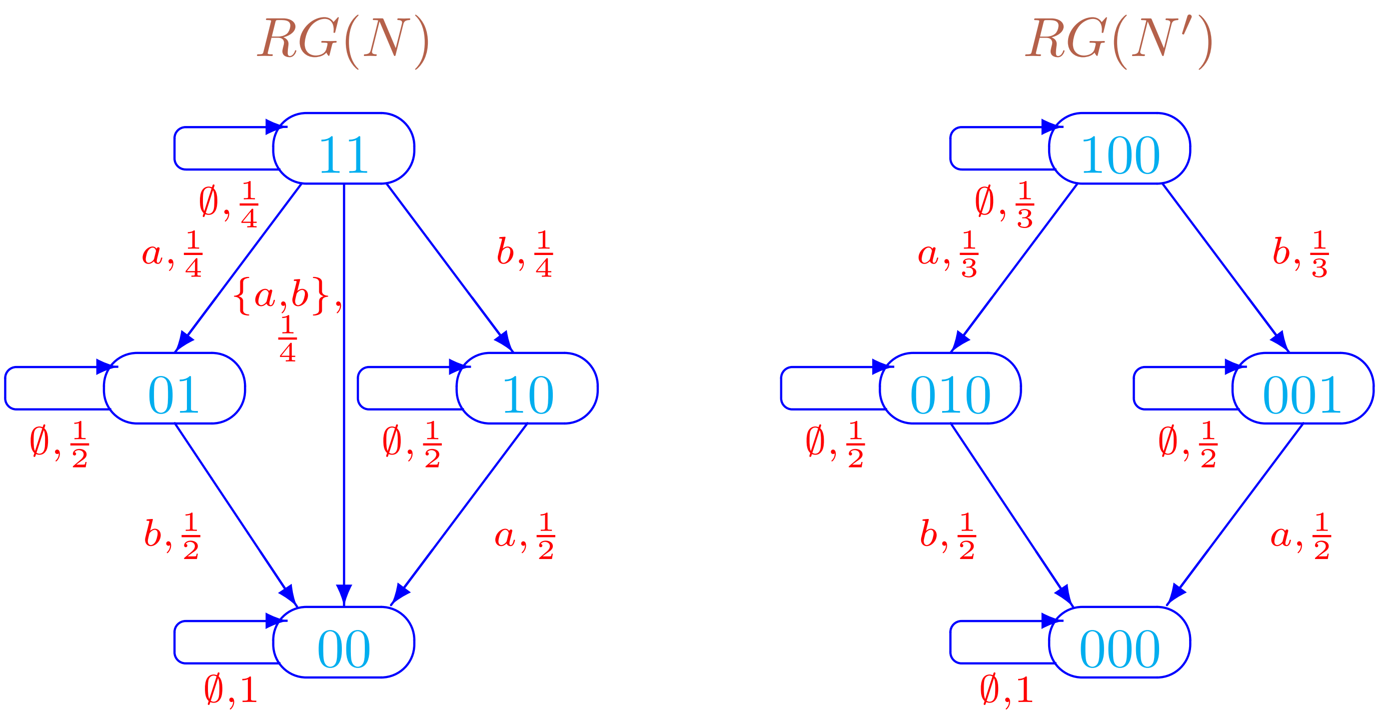
A LDTSPN N is a *image-finite* one, if
$\forall M \in RS^*(N)\, \forall A \in I\!N_f^{Act}\ |VisImage(M, A)| < \infty$.

Theorem 3 *For image-finite LDTSPNs N and N'*

$$N \underline{\leftrightarrow}_{sp}^{\tau} N' \ \Leftrightarrow\ N =_{SPML} N'.$$



Differentiating power of $=_{SPML}$



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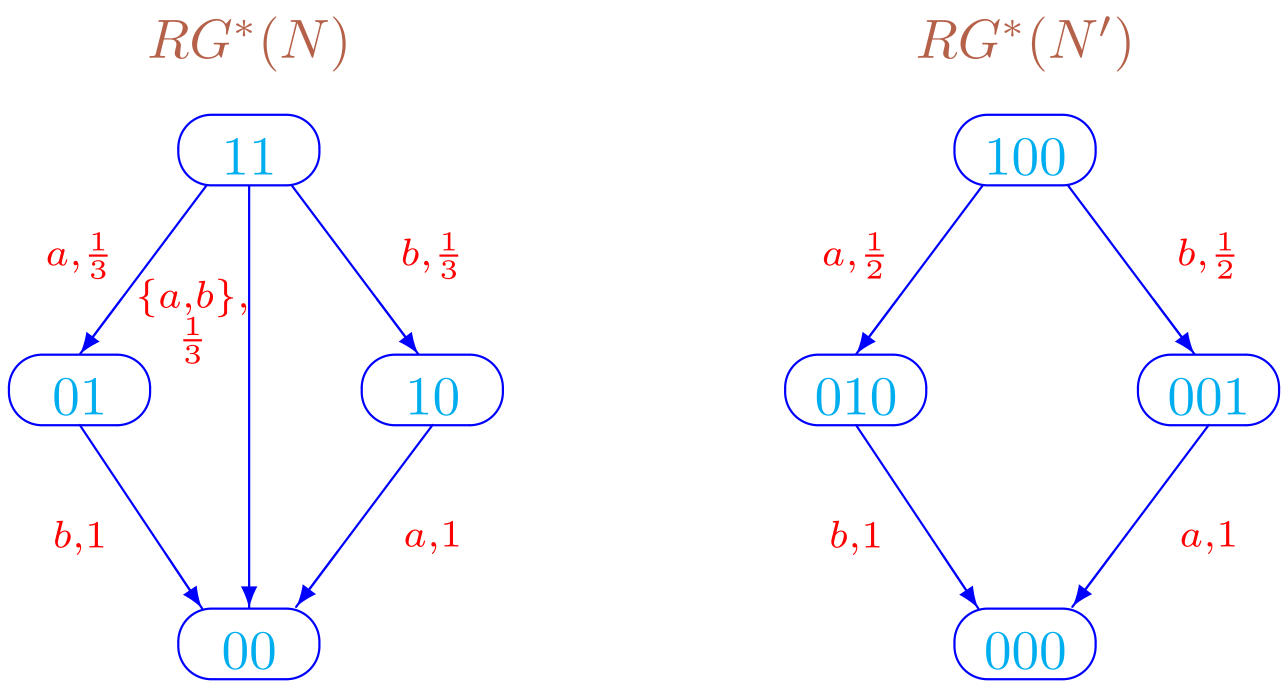


Visible reachability graphs of the LDTSPNs above

$N \underline{\leftrightarrow}^{\tau}_{ibfp} N'$ but $N \neq_{SPML} N'$, because for $\Phi = \langle\{a,b\}\rangle_{\frac{1}{3}}\top,\ N \models_N \Phi$, but $N' \not\models_{N'} \Phi$, since only in N' actions a and b cannot occur concurrently.

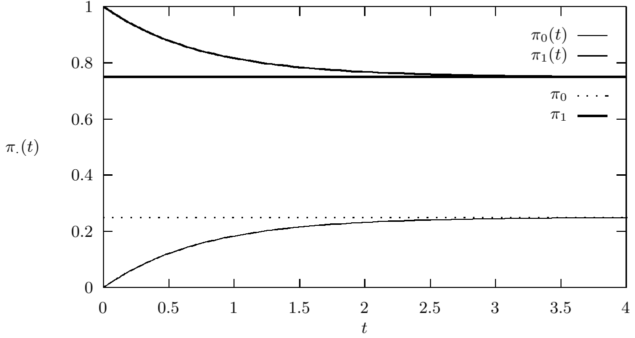


Fig. 6. Steady-state and transient behaviour of a 2-state CTMC (B.R. Haverkort, *Performance of Computer Communication Systems*, 1998. © John Wiley & Sons Limited. Reproduced with Permission.)

Stationary behavior

The *embedded steady state distribution* after the observation of a visible event is the unique solution of the equation system

$$\left\{ \begin{array}{l} \sum_{\widetilde{M}\in RS^*(N)} ps^*(\widetilde{M}) \cdot PM^*(\widetilde{M}, M) = ps^*(M) \\ \sum_{M\in RS^*(N)} ps^*(M) = 1 \end{array} \right. .$$

A *visible step probabilistic trace starting in* $M \in RS^*(N)$ is $(\Sigma, \mathcal{P})$, where $\Sigma = A_1 \cdots A_n \in Act^*$ and

$$\mathcal{P} = \sum_{\{M_1,\ldots,M_n \mid M \stackrel{A_1}{\twoheadrightarrow}_{\mathcal{P}_1} M_1 \stackrel{A_2}{\twoheadrightarrow}_{\mathcal{P}_2} \cdots \stackrel{A_n}{\twoheadrightarrow}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

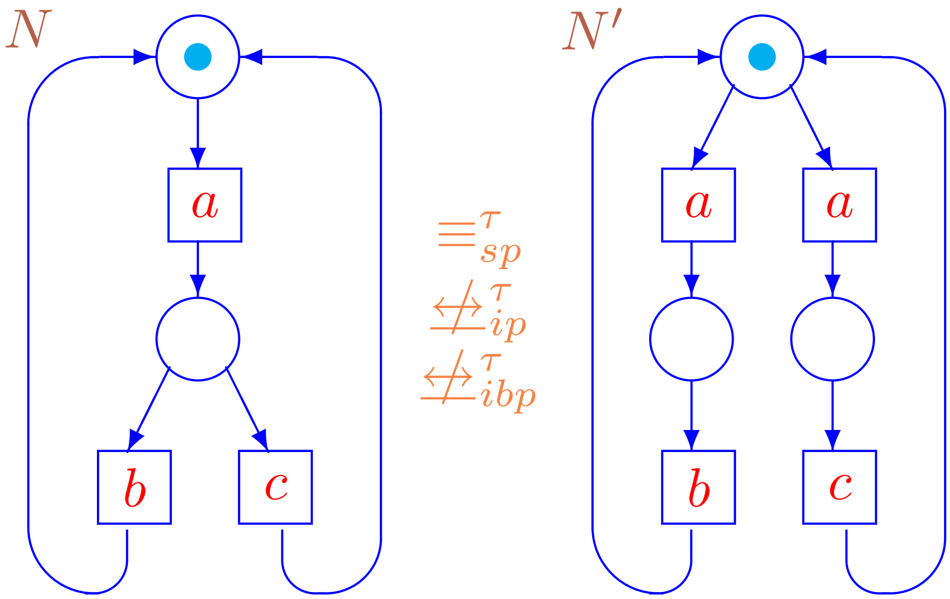
$VisStepProbTraces(N, M)$ is the set of *all visible step probabilistic traces starting in* $M \in RS^*(N)$.

Definition 8 *A* visible step probabilistic trace in steady state *is a triple* $(M, \Sigma, ps^*(M) \cdot \mathcal{P})$ *s.t* $M \in RS^*(N)$ *and* $(\Sigma, \mathcal{P}) \in VisStepProbTraces(N, M)$.

The set of all visible step probabilistic traces in steady state *is* $VisStepProbTracesSS(N)$.

Theorem 4 *Let for LDTSPNs* N *and* N' $N \underline{\leftrightarrow}_{sp}^{\tau} N'$ *or* $N \underline{\leftrightarrow}_{sbp}^{\tau} N'$. *Then*

$$VisStepProbTracesSS(N) = VisStepProbTracesSS(N').$$



SB: LDTSPNs with different visible step probabilistic traces in steady state

- In Figure SB, $N \equiv^{\tau}_{sp} N'$, but $VisStepProbTracesSS(N) \neq VisStepProbTracesSS(N')$.

  For N, the probability of being in one of both possible markings is $\frac{1}{2}$. Thus, a trace starts with a with probability $\frac{1}{2}$.

  For N', the probability of being in one of the three possible markings is $\frac{1}{3}$. Thus, a trace starts with a with probability $\frac{1}{3}$.

Solution methods for Markov chains [Hav01]

- Transient state probabilities
  - Runge-Kutta methods
  - Uniformisation
    (randomisation, Jensen's method): $O(\lambda t N)$ or $O(N^2)$
- Stationary state probabilities
  - Direct
    * Gaussian elimination: $O(N^3)$
    * LU decomposition: $O(N^3)$
  - Iterative
    * The power method: $O(N^2)$
    * The Jakobi method: $O(N^2)$
    * The Gauss-Seidel method: $O(N^2)$
    * The successive over-relaxation (SOR): $O(N^2)$

The results obtained

- A new class of stochastic Petri nets with labeled transitions and a step semantics for transition firing (LDTSPNs).
- Equivalences for LDTSPNs which preserve interesting aspects of behavior and thus can be used

  to compare systems and to compute for a given one a minimal equivalent representation [Buc95].
- A diagram of interrelations for the equivalences.
- Logical characterization of the equivalences via probabilistic modal logics.
- An application of the equivalences for comparing stationary behavior of LDTSPNs.

Further research

- Other equivalences in interleaving and step semantics:

  interleaving branching bisimulation [PRS92]

  (respecting conflicts with invisible transitions),

  back-forth bisimulations [NMV90,Pin93]

  (moving backward along history of computation).

- True concurrent equivalences:

  partial word and *pomset relations* [PRS92,Vog92,MCW03]

  (partial order models of computation).

- Logical characterization of *back and back-forth* equivalences:

  probabilistic extension of back-forth logic (BFL) [CLP92]

  (probabilistic eventuality operator for back moves).

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