Bull. Nov. Comp. Center, Comp. Science, 15 (2001), 31–64

Net and algebraic approaches to probabilistic modeling *

P. Buchholz, I. V. Tarasyuk

Abstract. This paper presents a class of Stochastic Petri Nets with concurrent transition firings. It is assumed that transitions occur in steps and for every step each enabled transition decides probabilistically whether it wants to participate in the step or not. Among the transitions which want to participate in a step, a maximal number is chosen to perform the firing step. The observable behavior of a net is described by labels associated with transitions. For this class of nets the dynamic behavior is defined and equivalence relations are introduced. These equivalences extend the well-known trace and bisimulation ones for systems with step semantics onto Stochastic Petri Nets with concurrent transition firing. It is shown that the equivalence notions form a lattice of interrelations. We demonstrate how the equivalences can be used to compare stationary behavior of nets. In addition, we propose a stochastic process algebra that describes a subclass of the nets we introduced.

Keywords: Stochastic Petri Nets, Step Semantics, Equivalence Relations, Bisimulation, Stationary Behavior, Stochastic Process Algebra.

1. Introduction

Stochastic Petri Nets (SPNs) are an established model type for the quantitative analysis of Discrete Event Dynamic Systems (DEDSs). SPNs were proposed about twenty years ago [11, 19] and are mainly considered on a continuous time scale which usually means that exponential or phase type distributions are associated with transitions. In this way, the stochastic process underlying an SPN is a Continuous Time Markov Chain (CTMC) that can be generated and analyzed by the well-known methods [24]. In SPNs of this class, only single transitions fire, so the well-known interleaving semantics is the basic approach for defining the dynamic behavior of SPNs. This interleaving behavior is also used in the case of Generalized Stochastic Petri Nets (GSPNs) [1, 7] which include timed transitions with exponential firing delay and immediate transitions with zero firing delay. Even for immediate transitions, interleaving semantics is usually considered. For SPNs

*Supported by post-doctoral scholarship of German Research Foundation (DFG) at Technical University of Dresden, by Russian Foundation for Basic Research under Grant 00-01-00898, by RAS Program "Projects of Young Scientists" under Grant 114, and by RAS Grant for Young Scientists.

and GSPNs, labeling of transitions has been recently introduced [4, 5]. After the definition of transition labeling, it is possible to define bisimulation equivalence for SPNs and GSPNs so that equivalent nets behave identically from the stochastic point of view. Details about the approach which introduces bisimulation for CTMCs with labeled transitions can be found in [3, 4, 13, 14].

Apart from continuous time distributions, discrete time distributions can also be assigned to transitions of Petri nets. Usually geometric distributions or mixtures of geometric distributions are used. The first approaches were published about 15 years ago [20], but more recent extensions of the basic class of nets with discrete time steps have also been proposed [27, 28]. To distinguish continuous and discrete time SPNs, we denote the former by CT-SPNs and the latter by DTSPNs. DTSPNs describe an underlying Discrete Time Markov Chain (DTMC). The major problem with this model class is that transitions fire concurrently, so that steps instead of interleavings are to be considered. This makes the interpretation and analysis of the model class more complex. For DTSPNs, labeling of transitions and an adequate definition of equivalence has not been introduced yet.

In this paper, we present an introduction of a new class of DTSPNs with labeled transitions. The dynamic behavior of this class of nets is characterized by steps instead of single transitions. The underlying stochastic process is still a DTMC, however, transitions of the DTMC describe sets of transitions that fire concurrently. Thus, commonly used notions defining bisimulation or trace equivalence of probabilistic processes [9, 18] are not adequate for this type of models.

Apart from SPNs, stochastic process algebras (SPAs) became very popular as a modeling framework in the recent years. Of particular interest is the relationship between SPNs and SPAs. In [8, 16], an Algebra of Finite nondeterministic Processes AFP_0 was proposed. Its formulas specify a special subclass of Petri nets: Acyclic or A-nets (ANs). We propose a stochastic extension of this calculus: an algebra $StAFP_0$ describing Stochastic A-nets (SANs). SANs are a subclass of DTSPNs. For a net equivalence (an isomorphism of net representations of algebraic formulas) we present a sound axiom system.

The outline of the rest of the paper is as follows. In Section 2, a new class of DTSPNs and the underlying stochastic process is introduced. Afterwards some examples are presented. Then, in Section 3, equivalence relations are defined for the presented class of nets, and interrelations between different equivalence relations are outlined. Section 4 briefly introduces the long run behavior of DTSPNs and describes which behavior is preserved by which equivalence relation. In Section 5, a Stochastic Algebra of Finite Processes $StAFP_0$ is proposed. In the concluding Section 6, we remind the main results of the paper and propose some directions of future research.

2. A class of discrete time stochastic Petri nets

In this section, we introduce the basic notions used throughout the paper and present several examples.

2.1. Formal definitions of the model and its behavior

DTSPNs which are the basic net class considered in this paper, are defined as follows.

Definition 1.
A *DTSPN* is a tuple $N = (P, T, W, \Lambda, \Omega, L, M_{in})$, where

- P and T are finite sets of *places* and *transitions*, respectively, such that $P \cup T \neq \emptyset$ and $P \cap T = \emptyset$;
- $W : (P \times T) \cup (T \times P) \to \mathbb{N}$ is a function describing the *weights of arcs* between places and transitions and vice versa;
- $\Lambda : T \to \mathbb{R}^+$ is the *transition weight* function;
- $\Omega : T \to (0, 1]$ is the *transition probability* function;
- $L : T \to Act_\tau$ is the *transition labeling* function assigning labels from a finite set of visible actions Act or an invisible action τ to transitions (i. e., $Act_\tau = Act \cup \{\tau\}$);
- $M_{in} : P \to \mathbb{N}$ is the *initial marking*.

The initial marking M_{in} is a specific case of a marking which assigns natural numbers to places. The marking of the net is modified by firing transitions. A transition $t \in T$ is enabled in a marking M if $M(p) \geq W(p, t)$ for all $p \in P$. Let $Ena(M)$ be the set (not a multiset) of *all transitions that are enabled in a marking M*. Firings of transitions are atomic operations, and transitions may fire concurrently. We assume that firings of transitions take place in steps. Since all transitions participating in a step should differ, we do not allow self-concurrency, i. e., firing of transitions concurrently to themselves.

A transition $t \in Ena(M)$ tries to fire in the next step with probability $\Omega(t)$. Let $U \subseteq Ena(M)$ be a set of transitions that try to fire in the next step. The probability that (all and only) the transitions from the set U decide to fire is given by

$$PF[U] = \prod_{t \in U} \Omega(t) \cdot \prod_{t \in Ena(M) \setminus U} (1 - \Omega(t)).$$

Probability $PF[\cdot]$ results from independent Bernoulli (binomial) trials of enabled transitions.

However, not necessarily the whole batch U can fire concurrently because transitions may be in conflict such that only a subset of U is able to fire. All transitions from a set U can fire if the following condition (*) holds:

$$\forall p \in P : M(p) \geq \sum_{t \in U} W(p,t).$$

If not all transitions from U can fire, then some maximal subset is chosen as shown below.

A set $V \subseteq Ena(M)$ is a *maximal fireable subset in a marking* M if (*) holds for V and no more transitions from $Ena(M) \setminus V$ can be added when the condition has to hold. By $MaxFire(M)$ we denote the set of *all maximal fireable subsets in a marking* M.

Similarly, a set $V \subseteq U$ is a *maximal fireable subset of* U *in a marking* M if (*) holds for V and no more transitions from $U \setminus V$ can be added when the condition has to hold. By $MaxFire(U,M)$ we denote the set of *all maximal fireable subsets of* U *in a marking* M.

We extend the weight function to the sets of transitions. If $V \subseteq T$ then $\Lambda(V) = \sum_{t \in V} \Lambda(t)$.

If transitions from the set U try to fire, but cannot fire concurrently since (*) does not hold, then one maximal fireable subset of transitions, i. e., one element from $MaxFire(U,M)$, is chosen. Subsets are chosen according to the normalized weights, i. e., weights Λ are essentially probabilistic means of solving the transition conflicts. Thus, a subset $V \in MaxFire(U,M)$ is chosen with probability

$$PC[V,U] = \Lambda(V) \Bigg/ \left(\sum_{W \in MaxFire(U,M)} \Lambda(W) \right).$$

For $V \subseteq U$ such that $V \notin MaxFire(U,M)$ we define $PC[V,U] = 0$, since we intend to avoid consideration of conflicting enabled transitions or not maximal sets of them.

For each $V \in MaxFire(M)$, let $SubEna(V,M)$ be the set of *all subsets of* $Ena(M)$ *that include* V. The probability of firing of $V \in MaxFire(M)$ is given by

$$PT[V,M] = \sum_{U \in SubEna(V,M)} PF[U] \cdot PC[V,U].$$

If no transition wants to fire at the next step, then $U = \emptyset = V$,
$PC[\emptyset, \emptyset] = 1$ and

$$PT[\emptyset, M] = PF[\emptyset] = \prod_{t \in Ena(M)} (1 - \Omega(t)).$$

The sets of enabled transitions that do not belong to $MaxFire(M)$ cannot fire concurrently in a marking M or they are not maximal and thus are of zero probability.

Note that $PF[\cdot]$ defines a probability distribution over subsets of $Ena(M)$. $PC[\cdot,\cdot]$ defines a conditional probability distribution over sets from $MaxFire(U, M)$.

We have not considered the labeling of transitions yet. The idea of labeling is that transitions receive the same label if they are indistinguishable for an external observer. We assume that the set of labels Act_τ contains a specific label τ that is not visible. Thus, transitions labeled with τ cannot be observed and called *invisible*.

Denote a *set of all finite multisets* over a set X by $\mathcal{M}(X)$.

We define the *visible labeling* function $VisL$ on the sets of transitions which associates the multisets of visible actions with them (note that we use summation instead of union, since we consider multisets). If $V \subseteq T$ then

$$VisL(V) = \sum_{(t \in V) \wedge (L(t) \neq \tau)} L(t).$$

Let $A \in \mathcal{M}(Act)$, i. e., A is a multiset of visible transition labels. Then

$$Trans(A) = \{V \subseteq T \mid VisL(V) = A\}$$

is the set of *all subsets of transitions which are labeled with* A. The probability of observing A in a marking M is then given by

$$PL[A, M] = \sum_{V \in Trans(A) \cap MaxFire(M)} PT[V, M].$$

Firing of sets of transitions yields a successor marking. If V fires in M, then the successor marking $\widetilde{M}$ is defined as

$$\widetilde{M}(p) = M(p) - \sum_{t \in V} W(p, t) + \sum_{t \in V} W(t, p).$$

Let V be a set of transitions which can fire concurrently in a marking M, the resulting marking be $\widetilde{M}$, and $\mathcal{P} = PT[V, M]$. We use the shortened notation $M \xrightarrow{V}_{\mathcal{P}} \widetilde{M}$ for such a firing step. We will write $M \xrightarrow{V} \widetilde{M}$ if $M \xrightarrow{V}_{\mathcal{P}} \widetilde{M}$ for some $\mathcal{P} > 0$. For a one-element set of transitions $V = \{t\}$, we write $M \xrightarrow{t}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{t} \widetilde{M}$.

By considering only the labels and not the concrete transitions, we obtain steps described by multisets of transition labels. Thus, $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ describes a step that starts in a marking M, performs transitions labeled with A and ends in $\widetilde{M}$. The probability of the step $\mathcal{P} = PS[A, M, \widetilde{M}]$ is

$$PS[A, M, \widetilde{M}] = \sum_{\{V \in Trans(A) | M \xrightarrow{V}_{\mathcal{Q}} \widetilde{M}\}} \mathcal{Q}.$$

We will write $M \xrightarrow{A} \widetilde{M}$ if $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ for some $\mathcal{P} > 0$. For a one-element multiset of actions $A = \{a\}$, we write $M \xrightarrow{a}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{a} \widetilde{M}$.

Definition 2. For a DTSPN N we define

- The *reachability set* $RS(N)$ as the minimal set of markings with the following conditions:
  - $M_{in} \in RS(N)$;
  - if $M \in RS(N)$ and $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ for $\mathcal{P} > 0$, then $\widetilde{M} \in RS(N)$.
- The *reachability graph* $RG(N)$ as a directed labeled graph with a set of nodes $RS(N)$ and an arc labeled with A, $\mathcal{P}$ between nodes M and $\widetilde{M}$ whenever $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ holds.
- The underlying *Discrete Time Markov Chain (DTMC)* $DT(N)$ with a state space $RS(N)$ and a transition $M \longrightarrow_{\mathcal{P}} \widetilde{M}$ whenever at least one arc between M and $\widetilde{M}$ exists in $RG(N)$. In this case, the probability $\mathcal{P} = PS[M, \widetilde{M}]$ is computed as

$$PS[M, \widetilde{M}] = \sum_{A \in \mathcal{M}(Act)} PS[A, M, \widetilde{M}].$$

The previous definition proposes the set of reachable markings, the corresponding reachability graph which preserves transition labels and probabilities and the underlying Discrete Time Markov Chain. Note that the reachability graph may include arcs with non-zero probability connecting different states such that the arcs correspond to the empty multisets. In this case, a marking is modified by firing some transition labeled with τ. It depends on the semantics of system dynamics or, better, on the underlying timing of the net whether an external observer can notice such a step or not. If we assume that time is slotted in such a way that an observer knows when a step takes place, then (s)he obviously notices that nothing happened, which means that an invisible step labeled with $\emptyset$ took place. However, if no such timing model exists, an observer cannot distinguish the situations that nothing happens or that an invisible step happened, i. e., an external observer who can only see visible transitions labeled with some action from *Act*. At the level of the DTMC, transition steps can no longer be distinguished, and we observe the stochastic process as a usual one for discrete time models like SPNs in discrete time [20, 27, 28]. Note that this DTMC with labeled transitions is different from probabilistic processes defined for probabilistic automata models like [15] where we have a labeling with sets of labels but no concurrently occuring transitions.

If we assume that an observer does not know when a step takes place, (s)he cannot see firing of a set of invisible transitions resulting in an empty multiset of transition labels. This behavior can be described by transforming the reachability graph by skipping unobservable transitions. This is the major viewpoint we take in this paper, but the following definitions of equivalences also hold for the case where all steps are observable. In the latter case, the following transformation of the reachability graph are not performed and equivalences are defined on $RG(N)$. The approach of transforming $RG(N)$ by skipping invisible transitions is similar to building the observational graph in untimed models [10].

An *internal step* $M \xrightarrow{\emptyset}_{\mathcal{P}} \widetilde{M}$ with $\mathcal{P} > 0$ takes place when $\widetilde{M}$ is reachable from M by firing of a set of invisible transitions or if no transition fires (in this case, $M = \widetilde{M}$). We use the following recursive definition of invisible transition probabilities:

$$PS^k[\emptyset, M, \widetilde{M}] = \begin{cases} \sum_{\overline{M} \in RS(N)} PS^{k-1}[\emptyset, M, \overline{M}] \cdot PS[\emptyset, \overline{M}, \widetilde{M}], & \text{if } k \geq 1; \\ 1, & \text{if } k = 0 \text{ and } M = \widetilde{M}; \\ 0, & \text{otherwise.} \end{cases}$$

$PS^k[\emptyset, M, \widetilde{M}]$ describes the *probability of reaching* $\widetilde{M}$ *from* M *by* k *internal steps*. We define

$$PS^*[\emptyset, M, \widetilde{M}] = \sum_{k=0}^{\infty} PS^k[\emptyset, M, \widetilde{M}]$$

which is the *probability of reaching* $\widetilde{M}$ *from* M *by internal steps* and

$$PS^*[A, M, \widetilde{M}] = \sum_{\overline{M} \in RS(N)} PS^*[\emptyset, M, \overline{M}] \cdot PS[A, \overline{M}, \widetilde{M}]$$

which is the *probability of reaching* $\widetilde{M}$ *from* M *by internal steps followed by an observable step* A.

We can define a new transition system with the transition relation $M \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PS^*[A, M, \widetilde{M}]$ and $A \neq \emptyset$.

We will write $M \xrightarrow{A}\!\!\!\!\rightarrow \widetilde{M}$ if $M \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}$ for some $\mathcal{P} > 0$. For a one-element multiset of actions $A = \{a\}$ we write $M \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{a}\!\!\!\!\rightarrow \widetilde{M}$.

We denote by $RS^*(N)$ and $RG^*(N)$ the *observable reachability set* and *graph*, respectively. Note that $RS(N) \neq RS^*(N)$, whenever there exist markings entered by invisible steps only (see also the examples given below). $RG^*(N)$ describes the viewpoint of a person who observes steps only if they include visible transitions.

We decided intentionally to consider only the sequences of internal steps followed by an observable step. Alternatively, we could consider an observable step followed by internal steps or an observable step preceded and succeeded by internal steps. In both cases our sequence ends with internal steps, which means that an observable reachability set (and graph) has to contain all the intermediate states which we go through while the suffix of internal steps occur. To avoid this complex description, we consider sequences ending with visible transitions.

We define an *embedded DTMC* $DT^*(N)$ with a state space $RS^*(N)$ and transition probabilities

$$PS^*[M, \widetilde{M}] = \sum_{A \in \mathcal{M}(Act) \setminus \emptyset} PS^*[A, M, \widetilde{M}].$$

A *trap* is a loop of invisible transitions starting and ending in some marking M which occurs with probability 1. If $RG(N)$ contains a trap, then the net sticks in a sequence of invisible transitions which cannot be left. The sum in the definition of $PS^*[\emptyset, M, \widetilde{M}]$ is finite as long as no traps exist which will be assumed in the sequel. In such a case, the definition of $PS^*[\emptyset, M, \widetilde{M}]$ makes sense and $PS^*[A, M, \widetilde{M}]$ defines a probability distribution, i. e.,

$$\sum_{A \in \mathcal{M}(Act) \setminus \emptyset} \sum_{\widetilde{M} \in RS^*(N)} PS^*[A, M, \widetilde{M}] = 1.$$

The result follows from the standard results on absorbing Markov chains [17].

Following the terminology of [12], we have introduced a generative model. However, in contrast to other stochastic models [9, 12, 18] which are based on some form of stochastic automata where only single events occur, we consider here the concurrent execution of *different* transitions. This is a very natural view for Petri nets which allow distributed state descriptions and parallel executions of transitions.

2.2. Examples of DTSPNs

The first example is shown in Figure 1. It describes a simple net with two observable transitions t_1 (labeled by a), t_2 (labeled by b) and one τ-labeled

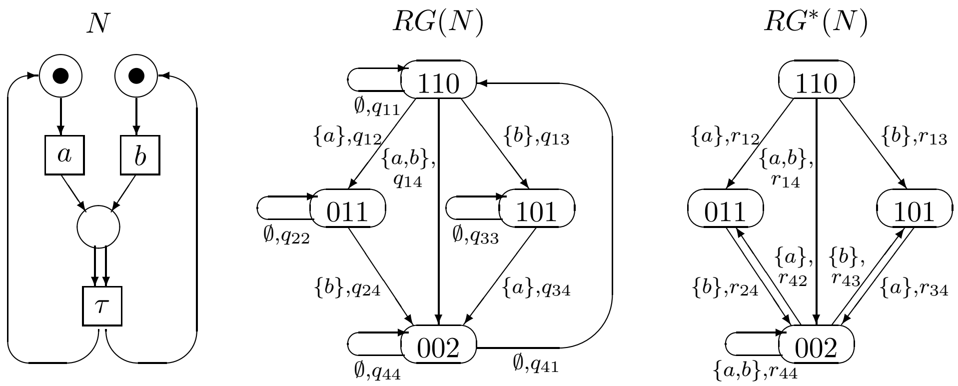

Figure 1. The first example: a net and the corresponding reachability graphs

transition t_3. The reachability graph $RG(N)$ and the observable reachability graph $RG^*(N)$ are also depicted in the figure. To define probabilities, we use the following numbering of markings: 1. (110), 2. (011), 3. (101), 4. (002). The values q_{ij} and r_{ij} are probabilities which receive the values shown below. The weights of transitions are not relevant in this example, because the net contains no conflict. For convenience we use the following notation:

$$\overline{\Omega}(t_i) = 1 - \Omega(t_i) \ (1 \le i \le 3).$$

Now we present the probabilities $q_{ij}(1 \le i, j \le 4)$:

$$\begin{array}{lll} q_{11} = \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_2) & q_{12} = \Omega(t_1) \cdot \overline{\Omega}(t_2) & q_{13} = \overline{\Omega}(t_1) \cdot \Omega(t_2) \\ q_{14} = \Omega(t_1) \cdot \Omega(t_2) & q_{22} = \overline{\Omega}(t_2) & q_{24} = \Omega(t_2) \\ q_{33} = \overline{\Omega}(t_1) & q_{34} = \Omega(t_1) & q_{41} = \Omega(t_3) \\ q_{44} = \overline{\Omega}(t_3) & & \end{array}$$

For the definition of r_{kl} $(1 \le k, l \le 4)$, the values q_{ij} defined above are used:

$$\begin{array}{lll} r_{12} = r_{42} = \frac{q_{12}}{1-q_{11}} & r_{13} = r_{43} = \frac{q_{13}}{1-q_{11}} & r_{14} = r_{44} = \frac{q_{14}}{1-q_{11}} \\ r_{24} = 1 & r_{34} = 1 & \end{array}$$

The second example is shown in Figure 2. It describes a net with two observable transitions t_1 (labeled by a), t_2 (labeled by b) and two τ-labeled transitions t_3 and t_4. To avoid an overloading of notations, if two arcs with different labels exist in $RG(N)$ or $RG^*(N)$, then only one arc is shown, and both labels are printed beneath the arc (i. e., $\{a\}, \{b\}$ describes that one arc labeled with $\{a\}$ and one arc $\{b\}$ are present). To define probabilities, we use the following numbering of markings: 1. (110), 2. (011), 3. (101),

4. (002), 5. (020) and 6. (200). Observe that $RS^*(N)$ contains only the markings 1–4. Markings 5 and 6 are not reachable, i. e., after an observable event, the net cannot be in one of these markings. We use the notation q_{ij}^A for the probability of the transition in $RG(N)$ between i and j which is labeled with the set A (for one-element multisets like $A = \{a\}$, we will omit the curly braces). If only one transition between i and j exists, then the label A is suppressed. Similarly r_{ij}^A is used for transition probabilities in $RG^*(N)$. To present the probabilities, we use the abbreviations

$$\Lambda_{34} = \frac{\Lambda(t_3)}{\Lambda(t_3) + \Lambda(t_4)} \text{ and } \Lambda_{43} = \frac{\Lambda(t_4)}{\Lambda(t_3) + \Lambda(t_4)}.$$

Thus, we obtain the probabilities q_{ij}^A $(1 \le i, j \le 6)$:

$$\begin{aligned}
q_{11} &= \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_2) \\
q_{12} &= \Omega(t_1) \cdot \overline{\Omega}(t_2) \\
q_{13} &= \overline{\Omega}(t_1) \cdot \Omega(t_2) \\
q_{14} &= \Omega(t_1) \cdot \Omega(t_2) \\
q_{21} &= \overline{\Omega}(t_2) \cdot \Omega(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4)) \\
q_{22}^{\emptyset} &= \overline{\Omega}(t_2) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4) \\
q_{22}^{b} &= \Omega(t_2) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3)) \\
q_{23} &= \Omega(t_2) \cdot \Omega(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4)) \\
q_{24} &= \Omega(t_2) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4) \\
q_{25} &= \overline{\Omega}(t_2) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3)) \\
q_{31} &= \overline{\Omega}(t_1) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3)) \\
q_{32} &= \Omega(t_1) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3)) \\
q_{33}^{\emptyset} &= \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4) \\
q_{33}^{a} &= \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4)) \\
q_{34} &= \Omega(t_1) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4) \\
q_{36} &= \overline{\Omega}(t_1) \cdot \Omega(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4)) \\
q_{41} &= \Omega(t_3) \cdot \Omega(t_4) \\
q_{42} &= \overline{\Omega}(t_3) \cdot \Omega(t_4) \\
q_{43} &= \Omega(t_3) \cdot \overline{\Omega}(t_4) \\
q_{42} &= \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4) \\
q_{52} &= \Omega(t_2) \\
q_{55} &= \overline{\Omega}(t_2) \\
q_{63} &= \Omega(t_1) \\
q_{66} &= \overline{\Omega}(t_1)
\end{aligned}$$

For the definition of probabilities r_{kl}^A $(1 \le k, l \le 4)$, we use the probabilities q_{ij}^A:

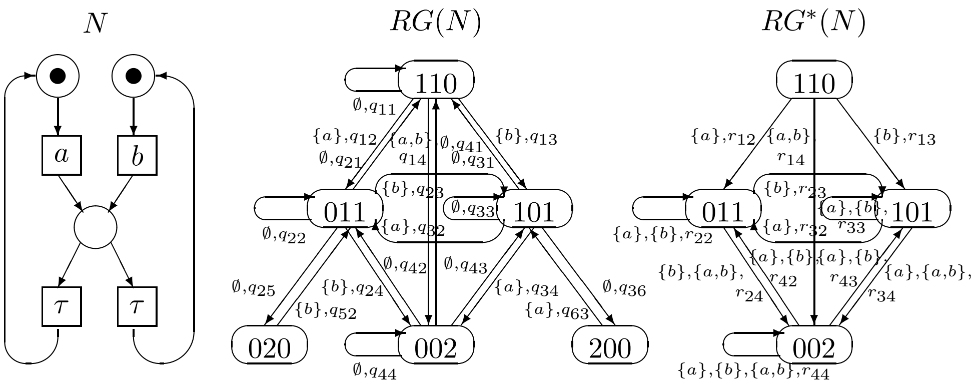

Figure 2. The second example: a net and the corresponding reachability graphs

$$
\begin{array}{ll}
r_{12} = q_{12}/(1 - q_{11}) & r_{13} = q_{13}/(1 - q_{11}) \\
r_{14} = q_{14}/(1 - q_{11}) & r_{22}^{a} = q_{21} \cdot r_{12}/(1 - q_{22}^{\emptyset}) \\
r_{22}^{b} = (q_{22}^{b} + q_{25})/(1 - q_{22}^{\emptyset}) & r_{23} = (q_{23} + q_{21} \cdot r_{13})/(1 - q_{22}^{\emptyset}) \\
r_{24}^{b} = q_{24}/(1 - q_{22}^{\emptyset}) & r_{24}^{\{a,b\}} = q_{21} \cdot r_{14}/(1 - q_{22}^{\emptyset}) \\
r_{32} = (q_{32} + q_{31} \cdot r_{12})/(1 - q_{33}^{\emptyset}) & r_{33}^{a} = (q_{33}^{a} + q_{36})/(1 - q_{33}^{\emptyset}) \\
r_{33}^{b} = q_{31} \cdot r_{13}/(1 - q_{33}^{\emptyset}) & r_{34}^{a} = q_{34}/(1 - q_{33}^{\emptyset}) \\
r_{34}^{\{a,b\}} = q_{31} \cdot r_{14}/(1 - q_{33}^{\emptyset}) & r_{42}^{a} = (q_{41} \cdot r_{12} + q_{43} \cdot r_{32})/(1 - q_{44}) \\
r_{42}^{b} = q_{42} \cdot r_{22}^{b}/(1 - q_{44}) & r_{43}^{a} = q_{43} \cdot r_{33}^{a}/(1 - q_{44}) \\
r_{43}^{b} = (q_{41} \cdot r_{13} + q_{42} \cdot r_{23})/(1 - q_{44}) & r_{44}^{a} = q_{43} \cdot r_{34}^{a}/(1 - q_{44}) \\
r_{44}^{b} = q_{42} \cdot r_{24}^{b}/(1 - q_{44}) & r_{44}^{\{a,b\}} = q_{41} \cdot r_{14}^{\{a,b\}}/(1 - q_{44})
\end{array}
$$

3. Equivalence relations for DTSPNs

Different equivalences have been proposed in the context of Petri nets [23, 25]. Furthermore, relations have been defined for probabilistic systems [9, 18]. However, in the probabilistic case, some sort of probabilistic interleaving is usually assumed so that only single transitions occur and not the sets of transitions. A widely used class of equivalence relations which have been defined in different settings are trace and bisimulation equivalences. Consequently, we propose the corresponding notions for DTSPNs.

3.1. Trace equivalences

Trace equivalences are the simplest ones. In the trace semantics, behavior of a system is associated with the set of all possible sequences of activities, i. e., protocols of work or computations. Thus, the points of choice of an external observer between several extensions of a particular computation are not taken into account.

Let us introduce the formal definitions of trace relations. These notions resemble those of trace relations for standard Petri nets from [25], but additionally have to take into account the probabilities of occurrences of sequences of (multisets of) actions. For this reason we have to collect probabilities of happening (multisets of) actions along *all possible* paths which correspond to our sequence in the observable reachability graphs $RG^*(N)$ and $RG^*(N')$ of two nets N and N' which are compared. Since we have already abstracted from particular transitions in such graphs, paths differ only in markings belonging to them. Thus, we should calculate a sum of probabilities for all paths according to a sequence of transition labels which differ at least in one marking.

Definition 3. An *interleaving trace* of a DTSPN N is a pair $(\sigma, \mathcal{P})$, where $\sigma = a_1 \cdots a_n \in Act^*$ and

$$\mathcal{P} = \sum_{\{M_1,\ldots,M_n | M_{in} \xrightarrow{a_1}\!\!\!\rightarrow_{\mathcal{P}_1} M_1 \xrightarrow{a_2}\!\!\!\rightarrow_{\mathcal{P}_2} \cdots \xrightarrow{a_n}\!\!\!\rightarrow_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

We denote a set of *all interleaving traces* of a DTSPN N by $IntTraces(N)$. Two DTSPNs N and N' are *interleaving trace equivalent*, denoted by $N \equiv_i N'$, if

$$IntTraces(N) = IntTraces(N').$$

Definition 4. A *step trace* of a DTSPN N is a pair $(\Sigma, \mathcal{P})$, where $\Sigma = A_1 \cdots A_n \in \mathcal{M}(Act)^*$ and

$$\mathcal{P} = \sum_{\{M_1,\ldots,M_n | M_{in} \xrightarrow{A_1}\!\!\!\rightarrow_{\mathcal{P}_1} M_1 \xrightarrow{A_2}\!\!\!\rightarrow_{\mathcal{P}_2} \cdots \xrightarrow{A_n}\!\!\!\rightarrow_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

We denote a set of *all step traces* of a DTSPN N by $StepTraces(N)$. Two DTSPNs N and N' are *step trace equivalent*, denoted by $N \equiv_s N'$, if

$$StepTraces(N) = StepTraces(N').$$

3.2. Bisimulation equivalences

Bisimulation equivalences completely respect points of choice of an external observer in the behavior of a modeled system, unlike the trace equivalences.

To define probabilistic bisimulation equivalences, we have to consider a bisimulation as an *equivalence* relation which partitions the states of the *union* of the observable reachability graphs $RG^*(N)$ and $RG^*(N')$ of two nets N and N' which are compared. For nets N and N' to be bisimulation equivalent, their initial markings M_{in} and M'_{in} have to be related by a bisimulation having the following transfer property: two markings are related if in each of them the same (multisets of) actions can occur, and the resulting markings *belong to the same equivalence class*. In addition, sums of probabilities for all such occurrences should be the same for both markings. Thus, in our definitions, we follow the approach of [18]. Hence, the difference between bisimulation and trace equivalences is that we do not consider *all possible* occurrences of (multisets of) actions from the initial markings, but only such that lead (stepwise) to markings *belonging to the same equivalence class*.

First we introduce several helpful notations. Let for a DTSPN N $\mathcal{L} \subseteq RS^*(N)$. For some $M \in RS^*(N)$ and $A \in \mathcal{M}(Act)$ we write $M \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L}$ if

$$\sum_{\{\widetilde{M} \in \mathcal{L} \mid M \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}\}} \mathcal{P} = \mathcal{Q}.$$

We will write $M \xrightarrow{A}\!\!\!\!\rightarrow \mathcal{L}$ if $M \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L}$ for some $\mathcal{Q} > 0$. For a one-element multiset of actions $A = \{a\}$ we write $M \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L}$ and $M \xrightarrow{a}\!\!\!\!\rightarrow \mathcal{L}$.

Let X be some set. The number of elements in X is denoted by $|X|$. We denote the cartesian product $X \times X$ by X^2. Let $\mathcal{E} \subseteq X^2$ be an equivalence relation on X. Then an *equivalence class* (w.r.t. $\mathcal{E}$) of an element $x \in X$ is defined by $[x]_{\mathcal{E}} = \{y \in X \mid (x, y) \in \mathcal{E}\}$. The equivalence $\mathcal{E}$ partitions X in the *set of equivalence classes* $X/_{\mathcal{E}} = \{[x]_{\mathcal{E}} \mid x \in X\}$.

Definition 5. Let N be a DTSPN. An *equivalence* relation $\mathcal{R} \subseteq RS^*(N)^2$ is *interleaving bisimulation* between two markings M_1 and M_2 of N (i. e., $(M_1, M_2) \in \mathcal{R}$), denoted by $\mathcal{R} : M_1 \underline{\leftrightarrow}_i M_2$, if $\forall a \in Act\ \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$

$$M_1 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L} \;\Leftrightarrow\; M_2 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L}.$$

Two markings M_1 and M_2 are *interleaving bisimulation equivalent*, denoted by $M_1 \underline{\leftrightarrow}_i M_2$, if $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}_i M_2$.

To introduce bisimulation between two DTSPNs N and N', we should consider a “composite” set or reachable states, i. e., $RS^*(N) \cup RS^*(N')$.

Definition 6. Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is *interleaving bisimulation* between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}_i N'$, if $\mathcal{R} : M_{in} \underline{\leftrightarrow}_i M'_{in}$.

Two DTSPNs N and N' are *interleaving bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_i N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_i N'$.

Definition 7. Let N be a DTSPN. An *equivalence* relation $\mathcal{R} \subseteq RS^*(N)^2$ is *step bisimulation* between two markings M_1 and M_2 of N, denoted by $\mathcal{R} : M_1 \underline{\leftrightarrow}_s M_2$, if $\forall A \in \mathcal{M}(Act)\ \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$

$$M_1 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L} \ \Leftrightarrow\ M_2 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L}.$$

Two markings M_1 and M_2 are *step bisimulation equivalent*, denoted by $M_1 \underline{\leftrightarrow}_s M_2$, if $\exists \mathcal{R} :\ M_1 \underline{\leftrightarrow}_s M_2$.

Definition 8. Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is *step bisimulation* between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}_s N'$, if $\mathcal{R} : M_{in} \underline{\leftrightarrow}_s M'_{in}$.

Two DTSPNs N and N' are *step bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_s N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_s N'$.

It is easy to show that the union of two (interleaving or step) bisimulations is also (interleaving or step) bisimulation such that the largest bisimulation relation exists and is unique up to the ordering of equivalence classes. Consequently, for a given DTSPN, equivalent nets with a minimal state space exist.

3.3. Backward bisimulation equivalences

For untimed systems apart from bisimulation in forward direction, bisimulation in backward direction has also been defined [21, 22]. However, the definition introduced in [21] is not a straightforward extension of forward bisimulation which would simply mean to define backward bisimulation as bisimulation on the transition graph after reversing the direction of arcs. The authors in [21] argue why such a definition is not useful in their context of untimed systems and define backward bisimulation based on paths preserving the history that brought the system to a state. This definition cannot be transferred to our viewpoint of stochastic systems. Instead we define here backward bisimulation by extending forward bisimulation using two additional conditions on the initial marking and on outgoing transition probabilities. The latter implies that we define some form of back and forth bisimulation. However, we use the notation *backward bisimulation* for the resulting equivalence which has shown to be useful for stochastic automata networks [6] and can be transferred naturally to DTSPNs.

Like bisimulation, which will from now on also be denoted by *forward bisimulation*, backward bisimulation is defined using equivalence relations. For $\mathcal{L} \subseteq RS^*(N),\ M \in RS^*(N)$ and $A \in \mathcal{M}(Act)$ we define $\mathcal{L} \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} M$ as

$$\sum_{\{\widetilde{M}\in\mathcal{L}|\widetilde{M}\xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}}M\}} \mathcal{P} = \mathcal{Q}.$$

We will write $\mathcal{L} \xrightarrow{A}\!\!\!\!\rightarrow M$ if $\mathcal{L} \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} M$ for some $\mathcal{Q} > 0$. For a one-element multiset of actions $A = \{a\}$, we write $\mathcal{L} \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} M$ and $\mathcal{L} \xrightarrow{a}\!\!\!\!\rightarrow M$.

Definition 9. Let N be a DTSPN. An *equivalence* relation $\mathcal{R} \subseteq RS^*(N)^2$ is an *interleaving backward bisimulation* between two markings M_1 and M_2 of N, denoted by $\mathcal{R} : M_1 \underline{\leftrightarrow}_{ib} M_2$, if $\forall a \in Act\ \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$

$$M_1 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(N) \ \Leftrightarrow\ M_2 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(N),$$
$$\mathcal{L} \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} M_1 \ \Leftrightarrow\ \mathcal{L} \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} M_2 \text{ and } [M_{in}]_{\mathcal{R}} = \{M_{in}\}.$$

Two markings M_1 and M_2 are *interleaving backward bisimulation equivalent*, denoted by $M_1 \underline{\leftrightarrow}_{ib} M_2$, if $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}_{ib} M_2$.

Observe that backward bisimulation has a part looking forward in the future due to identical probability sums of leaving a marking via a-labeled transitions and a part looking backward due to identical probabilities of incoming transitions from each equivalence class. The definition of backward bisimulation for two nets looks a little bit more complicated than the corresponding definition for forward bisimulation, because we cannot assume that incoming transition probabilities are the same for equivalent markings from different nets. Instead it should be assured that the probability flow from one equivalence class to another is the same in both nets, and for each net the flow into each marking of an equivalence class should be the same. To simplify the definitions mentioned here, we propose the following *indicator function* Γ which recovers a DTSPN by a marking belonging to it. Let N be a DTSPN and $M \in RS^*(N)$, then $\Gamma(M) = N$. Thus, this is just a convenient notation allowing one to avoid treatment of different cases when markings of two nets are considered together.

Definition 10. Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is *interleaving backward bisimulation* between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}_{ib} N'$, if $\forall a \in Act\ \forall \mathcal{L}, \mathcal{K} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}\ \forall M_1, M_2 \in \mathcal{L}$

$$M_1 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(\Gamma(M_1)) \ \Leftrightarrow\ M_2 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(\Gamma(M_2)),\ [M_{in}]_{\mathcal{R}} = \{M_{in}, M'_{in}\}$$

and

$$\mathcal{K} \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_1))|}{|\mathcal{K}\cap RS^*(\Gamma(M_1))|}} M_1 \ \Leftrightarrow\ \mathcal{K} \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_2))|}{|\mathcal{K}\cap RS^*(\Gamma(M_2))|}} M_2.$$

Two DTSPNs N and N' are *interleaving backward bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_{ib} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_{ib} N'$.

For markings M_1 and M_2 belonging to the same net, the conditions on incoming probabilities reduce to the requirement of identical incoming probabilities.

Definition 11. Let N be a DTSPN. An *equivalence* relation $\mathcal{R} \subseteq RS^*(N)^2$ is *step backward bisimulation* between two markings M_1 and M_2 of N, denoted by $\mathcal{R} : M_1 \underline{\leftrightarrow}_{sb} M_2$, if $\forall A \in \mathcal{M}(Act)\ \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$

$$M_1 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(N) \ \Leftrightarrow\ M_2 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(N),\ \mathcal{L} \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} M_1 \ \Leftrightarrow\ \mathcal{L} \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} M_2$$

and

$$[M_{in}]_{\mathcal{R}} = \{M_{in}\}.$$

Two markings M_1 and M_2 are *step backward bisimulation equivalent*, denoted by $M_1 \underline{\leftrightarrow}_{sb} M_2$, if $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}_{sb} M_2$.

Definition 12. Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is *step backward bisimulation* between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}_{sb} N'$, if $\forall A \in \mathcal{M}(Act)\ \forall \mathcal{L}, \mathcal{K} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}\ \forall M_1, M_2 \in \mathcal{L}$

$$M_1 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(\Gamma(M_1)) \ \Leftrightarrow\ M_2 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(\Gamma(M_2)),\ [M_{in}]_{\mathcal{R}} = \{M_{in}, M'_{in}\}$$

and

$$\mathcal{K} \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_1))|}{|\mathcal{K}\cap RS^*(\Gamma(M_1))|}} M_1 \ \Leftrightarrow\ \mathcal{K} \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_2))|}{|\mathcal{K}\cap RS^*(\Gamma(M_2))|}} M_2.$$

Two DTSPNs N and N' are *step backward bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_{sb} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_{sb} N'$.

As before, the union of backward bisimulations is backward bisimulation.

3.4. Back and forth bisimulation equivalences

A natural way to define a new equivalence is to combine backward and forward bisimulation. We define here only back and forth bisimulation equivalences for two nets, the remaining definitions can be transferred similarly. As before, the notions of interleaving and step equivalences are proposed.

Definition 13. Two DTSPNs N and N' are *interleaving back and forth bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_{ibf} N'$, if $N \underline{\leftrightarrow}_{i} N'$ and $N \underline{\leftrightarrow}_{ib} N'$.

Definition 14. Two DTSPNs N and N' are *step back and forth bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_{sbf} N'$, if $N \underline{\leftrightarrow}_{s} N'$ and $N \underline{\leftrightarrow}_{sb} N'$.

3.5. Examples of the equivalences

Let us present some examples of equivalence relations.

As we have already seen, one can consider bisimulation between a net and itself, i. e., bisimulation between markings of the net and bisimulation between different nets. Let us first consider equivalence of markings of a single net for the net shown in Figure 1. Markings (110) and (002) of N are forward bisimilar, if $r_{12} = r_{42}$, $r_{13} = r_{43}$ and $r_{44} = r_{14}$, which holds by definition of the transition probabilities. If we assume that a and b are identical symbols, then (011) and (101) are forward bisimulation equivalent independently of $\Lambda(t_1)$ and $\Lambda(t_2)$, as long as both values are non-zero, which has been assumed when $RS^*(N)$ has been generated. Observe that this bisimulation is not backward bisimulation.

For bisimulation between different nets we consider the example shown in Figure 3. We assume that conflicting transitions have the same weights and firing probabilities. All nets have a very simple structure without concurrently enabled transitions such that interleaving behavior is identical to the step one.

The following equivalence relations exist between the nets:

$$N_1 \equiv_s N_2 \equiv_s N_3 \equiv_s N_4 \quad N_1 \underline{\leftrightarrow}_s N_2 \underline{\leftrightarrow}_s N_4 \quad N_1 \underline{\leftrightarrow}_{sb} N_3 \underline{\leftrightarrow}_{sb} N_4 \quad N_1 \underline{\leftrightarrow}_{sbf} N_4.$$

Observe that there is no bisimulation relation between N_2 and N_3, i. e., $N_2 \not\underline{\leftrightarrow}_i N_3$ and $N_2 \not\underline{\leftrightarrow}_{ib} N_3$.

3.6. Interrelations between equivalences

In this section, we compare the introduced equivalences and obtain the lattice of their interrelations.

Proposition 1. *Let $\star \in \{i, s\}$. For DTSPNs N and N' the following holds:*

$$N \underline{\leftrightarrow}_\star N' \ \Rightarrow \ N \equiv_\star N'.$$

Proof. See Appendix I. □

In a similar way, we show that backward bisimulation implies trace equivalence.

Proposition 2. *Let $\star \in \{i, s\}$. For DTSPNs N and N' the following holds:*

$$N \underline{\leftrightarrow}_{\star b} N' \ \Rightarrow \ N \equiv_\star N'.$$

Proof. See Appendix II. □

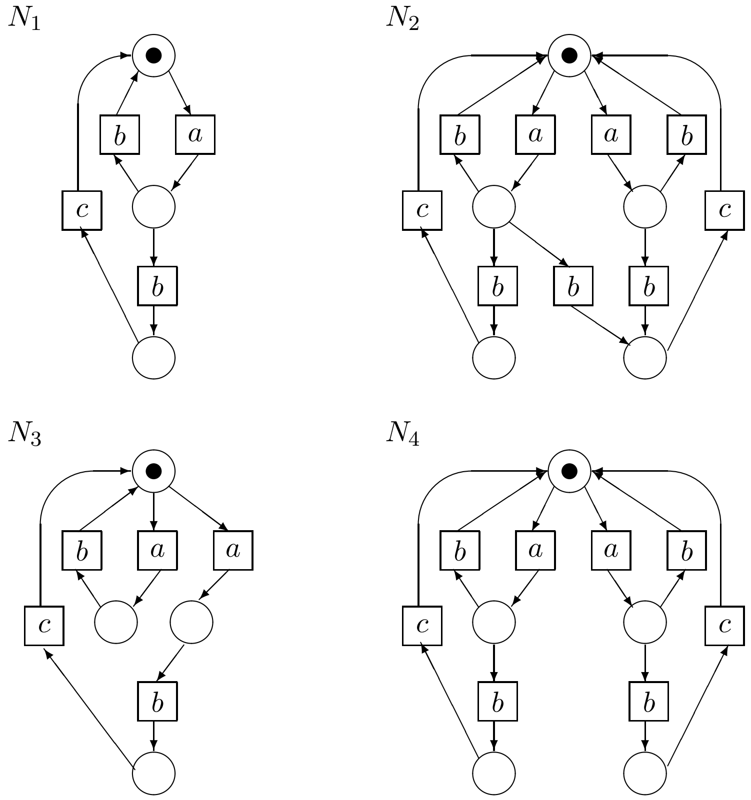

Figure 3. Nets related via different equivalences

The following proposition concerns relations of back and forth bisimulations with the other ones.

Proposition 3. *Let $\star \in \{i, s\}$. For DTSPNs N and N' the following holds:*

$$N \underline{\leftrightarrow}_{\star bf} N' \;\Rightarrow\; N \underline{\leftrightarrow}_{\star} N' \text{ and } N \underline{\leftrightarrow}_{\star b} N'.$$

Proof. The result follows from the definitions of back and forth bisimulations. □

Thus, we have obtained several important results for our equivalences stating that bisimulation (forward or backward) relations imply trace ones. This helps us to establish interrelations of the introduced equivalence notions.

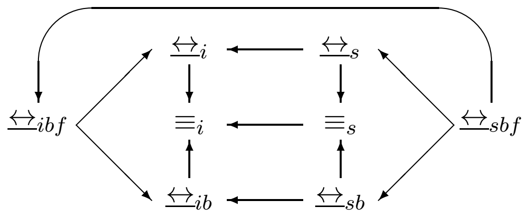

Figure 4. Interrelations between equivalences

Theorem 1. *Let* $\leftrightarrow, \twoheadleftarrow\!\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$ *and* $\star, \star\star \in \{i, s, ib, sb, ibf, sbf\}$. *For DTSPNs* N *and* N' *the following holds:*

$$N \leftrightarrow_\star N' \;\Rightarrow\; N \twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star} N'$$

iff in the graph in Figure 4 there exists a directed path from $\leftrightarrow_\star$ *to* $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star}$.

Proof. ($\Leftarrow$) Let us check the validity of implications in the graph in Figure 4.

- The implications $\leftrightarrow_s \to \leftrightarrow_i$, $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$, and $\underline{\leftrightarrow}_{sb} \to \underline{\leftrightarrow}_{ib}$, $\underline{\leftrightarrow}_{sbf} \to \underline{\leftrightarrow}_{ibf}$, are valid since actions are one-element multisets.
- The implications $\underline{\leftrightarrow}_\star \Rightarrow \equiv_\star$, $\underline{\leftrightarrow}_{\star b} \Rightarrow \equiv_\star$, $\star \in \{i, s\}$, are valid by Proposition 1 and Proposition 2, respectively.
- The implications $\underline{\leftrightarrow}_{\star bf} \Rightarrow \underline{\leftrightarrow}_\star$, $\underline{\leftrightarrow}_{\star bf} \Rightarrow \underline{\leftrightarrow}_{\star b}$, $\star \in \{i, s\}$, are valid by Proposition 3.

($\Rightarrow$) Absence of additional nontrivial arrows in the graph in Figure 4 is proved by the following examples. As in the previous examples, we assume that conflicting transitions have equal weights and probabilities.

- In Figure 5(a), $N \underline{\leftrightarrow}_{ibf} N'$ but $N \not\equiv_s N'$, since only in the DTSPN N' actions a and b cannot happen concurrently.
- In Figure 5(b), $N \equiv_s N'$ but $N \underline{\not\leftrightarrow}_i N'$ and $N \underline{\not\leftrightarrow}_{ib} N'$, since only in the DTSPN N' an action a can happen so that no action b can happen afterwards.
- In Figure 3, $N_1 \underline{\leftrightarrow}_s N_2$ but $N_1 \underline{\not\leftrightarrow}_{ib} N_2$, since only in N_2 there is a place with two input transitions labeled by b. Hence, the probability for a token to go to this place is always more than for that with only one input b-labeled transition.
- In Figure 3, $N_1 \underline{\leftrightarrow}_{sb} N_3$ but $N_1 \underline{\not\leftrightarrow}_i N_3$, since only in the DTSPN N_1 an action a can happen so that a sequence of actions bc cannot happen just after it. □

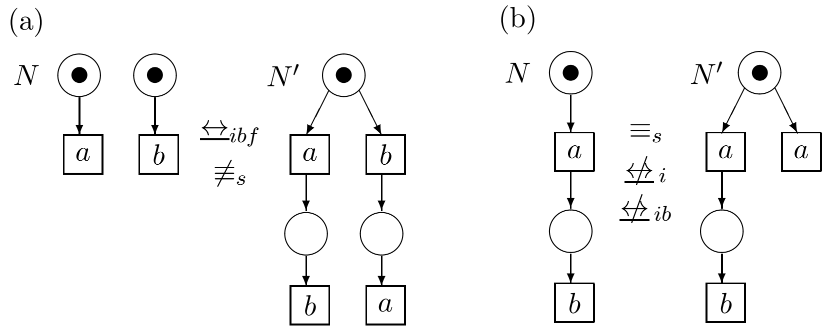

Figure 5. Examples of the equivalences

4. Stationary behavior of DTSPNs

A natural observation of the behavior of a dynamic system is the observation of traces starting from the initial marking of the DTSPN. Depending on the chosen viewpoint, steps or only single transitions are observed. Traces have been used to define trace equivalence. Consequently, trace equivalent DTSPNs have the same traces, and since trace equivalence is the weakest relation we have defined, all other equivalences also preserve traces.

An alternative and commonly used viewpoint in stochastic systems is to consider the DTSPN in its steady state. For this behavior we consider only nets with an infinite behavior and assume that the embedded DTMC is irreducible or contains at least only one irreducible subset of markings. The embedded steady state distribution after the observation of a visible event is the unique solution of the set of linear equations

$$ps^*(M) = \sum_{\widetilde{M} \in RS^*(N)} ps^*(\widetilde{M}) \cdot PS^*[\widetilde{M}, M]$$

subject to $\sum_{M \in RS^*(N)} ps^*(M) = 1$.

Further we consider only step behavior but the results can be easily formulated for interleaving behavior as well. First, extend the notion of step traces by defining a *step trace starting in a marking* $M \in RS^*(N)$ as $(\Sigma, \mathcal{P})$, where $\Sigma = A_1 \cdots A_n \in Act^*$ and

$$\mathcal{P} = \sum_{\{M_1,\ldots,M_n | M \xrightarrow{A_1}_{\mathcal{P}_1} M_1 \xrightarrow{A_2}_{\mathcal{P}_2} \cdots \xrightarrow{A_n}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

Thus, in the definition of $StepTraces(N)$ we replace M_{in} by M. Let $StepTraces(N, M)$ be the set of *all step traces* of DTSPN N *starting in a marking* M.

Definition 15. A *step trace in steady state* is a triple $(M, \Sigma, ps^*(M) \cdot \mathcal{P})$ s.t. $M \in RS^*(N)$ and $(\Sigma, \mathcal{P}) \in StepTraces(N, M)$. The set of *all step traces in steady state* is denoted by $StepTracesSS(N)$.

Now we show that forward or backward bisimulation equivalent nets have the same steady state traces, whereas trace equivalence does not preserve steady state traces.

Proposition 4.

1. *Let N and N' be two forward bisimulation equivalent DTSPNs, then* $\forall \mathcal{L} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}$

$$\sum_{M \in \mathcal{L} \cap RS^*(N)} ps^*(M) = \sum_{M' \in \mathcal{L} \cap RS^*(N')} ps^*(M').$$

2. *Let N and N' be two backward bisimulation equivalent DTSPNs, then* $\forall \mathcal{L} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}$

$$\sum_{M \in \mathcal{L} \cap RS^*(N)} ps^*(M) = \sum_{M' \in \mathcal{L} \cap RS^*(N')} ps^*(M');$$

$\forall M, \widetilde{M} \in \mathcal{L} \cap RS^*(N),\ \forall M', \widetilde{M}' \in \mathcal{L} \cap RS^*(N')$

$$ps^*(M) = ps^*(\widetilde{M}) \text{ and } ps^*(M') = ps^*(\widetilde{M}').$$

Proof. The proof is an extension of the corresponding results for the continuous time case [3, 4]. □

Theorem 2. *Let N and N' be backward or forward bisimulation equivalent DTSPNs, then*

$$StepTracesSS(N) = StepTracesSS(N').$$

Proof. See Appendix III. □

The implication stated in the previous theorem cannot be reversed, since for step trace equivalent nets N and N', we may have $StepTracesSS(N) \neq StepTracesSS(N')$. This can be seen from the two nets shown in Figure 6. For the net N, the probability of being in one of both possible markings is 1/2. Consequently, a trace starts with a with probability 1/2. For the net N', the probability of being in one of the three possible markings after

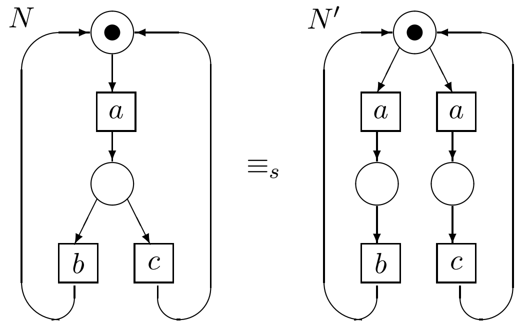

Figure 6. Two step trace equivalent nets with $StepTracesSS(N) \neq StepTracesSS(N')$

observation of a transition equals 1/3. Consequently, the probability of observing a trace starting with a equals 1/3.

One should note that the stationary distribution is defined here according to the embedded distribution after observing a step of visible transitions. This distribution differs from the stationary distribution of the net at an arbitrary time. The latter behavior has to be analyzed on $RS(N)$ instead of $RS^*(N)$ and is not preserved by any of the proposed equivalences even if we restrict the observation to visible transitions.

5. Stochastic process algebra $StAFP_0$

In [8, 16], an Algebra of Finite nondeterministic parallel Processes AFP_0 was proposed. Its formulas specify Acyclic nets (ANs). We propose a stochastic extension of this calculus, *Stochastic Algebra of Finite Processes* $StAFP_0$ describing Stochastic A-nets (SANs), i. e., ANs with transition probabilities.

5.1. Syntax

An *activity* is a pair (a, ω), where $a \in Act$ is the action label and $\omega \in (0, 1]$ is the probability of action a. Let AP be the set of *all activities.*

Activities are combined into formulas by the following operations: *concurrency* $\|$, *precedence* ; and *alternative* $\bigtriangledown$.

Definition 16. Let $(a, \omega) \in AP$. A *formula* of $StAFP_0$ is defined as follows:

$$E ::= \ (a, \omega) \mid E\|E \mid E; E \mid E \bigtriangledown E.$$

$\mathbf{StAFP_0}$ denotes the set of *all formulas* of $StAFP_0$.

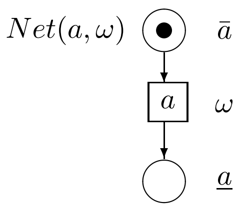

Figure 7. An atomic net

5.2. Semantics

Let N be a DTSPN. Then $^\circ N = \{p \in P \mid \forall t \in T\ W(t,p) = \emptyset\}$ is the set of *initial* places of N and $N^\circ = \{p \in P \mid \forall t \in T\ W(p,t) = \emptyset\}$ is the set of *final* places of N.

As we will see, formulas of $StAFP_0$ specify a subclass of DTSPNs with equal transition weights (e.g., all weights are equal to 1) and the initial marking coinciding with the set of initial places. In addition, the underlying Petri nets are ANs. This means that nets are strictly labeled by actions from *Act* (there are no invisible transitions) and all transitions have different labels. We call this subclass *Stochastic A-nets (SANs).*

In the specification of SANs, we will omit the transition weight function, since $\Lambda = 1$, and the initial marking, since $M_{in} = {}^\circ N$. Due to strict labeling of ANs, for SANs we can suppose $T \subseteq Act$, and thus we can omit labeling function, since $L = id_T$. Hence, a SAN can be specified by a quadruple $N = (P, T, W, \Omega)$.

Now we introduce a mapping Net from $\mathbf{StAFP_0}$ to SANs.

Let $(a, \omega) \in AP$. An *atomic net* $Net(a, \omega)$ consists of a transition with label a and probability ω having one initial and one final place connected with the transition by ordinary arcs. The initial place contains one token. Thus, $Net(a, \omega) = (P, T, W, \Omega)$, where

- $P = \{\bar{a}, \underline{a}\}$;
- $T = \{a\}$;
- $W = \{(\bar{a}, a), (a, \underline{a})\}$;
- $\Omega = \{(a, \omega)\}$.

An example of atomic net is presented in Figure 7.

To define the mapping Net for composed formulas, we need some additional notions.

First, for a SAN $N = (P, T, W, \Omega)$, we propose a *forming* operation $\otimes$ over two sets of its places $Q, R \subseteq P$:

$$R \otimes Q = \{r \cup q \mid r \in R,\ q \in Q\}.$$

The *merging* operation μ over a SAN $N = (P, T, W, \Omega)$ merges two sets of its places $Q, R \subseteq P$:

$$\mu(N, R, Q) = (\widetilde{P}, T, \widetilde{W}, \Omega),$$

where

- $\widetilde{P} = P \setminus (R \cup Q) \cup (R \otimes Q)$;
- $\widetilde{W}(p) = \begin{cases} W(p), & p \in \widetilde{P} \setminus (R \otimes Q); \\ W(r) \cup W(q), & p = (r \cup q) \in R \otimes Q,\ r \in R,\ q \in Q. \end{cases}$

Let $N = (P, T, W, \Omega)$ and $N' = (P', T', W', \Omega')$ be two SANs. We define net operations as follows.

Concurrency $N \| N' = (P \cup P', T \cup T', W \cup W', \widetilde{\Omega})$, where

$$\widetilde{\Omega}(a) = \begin{cases} \Omega(a), & a \in T \setminus T'; \\ \Omega'(a), & a \in T' \setminus T; \\ \Omega(a) \cdot \Omega'(a), & a \in T \cap T'. \end{cases}$$

Precedence $N; N' = \mu(N \| N', N^\circ, {}^\circ N')$.

Alternative $N \bigtriangledown N' = \mu(\mu(N \| N', {}^\circ N, {}^\circ N'), N^\circ, N'^\circ)$.

As one can see, by concurrent composition we synchronize processes containing the same actions. In this case, the probabilities of actions to be merged are multiplied.

We suppose that SANs N and N' combined by net operations ; and $\bigtriangledown$ do not contain equally named transitions. In the case of formulas, we suppose that P and P' combined by corresponding formula operations ; and $\bigtriangledown$ contain no identical actions. In any case, it is always possible to rename the actions and recover the names after applying the operations.

Let $E, F \in \mathbf{StAFP_0}$. We define the mapping Net as follows:

1. $Net(E \| F) = Net(E) \| Net(F)$;
2. $Net(E; F) = Net(E); Net(F)$;
3. $Net(E \bigtriangledown F) = Net(E) \bigtriangledown Net(F)$.

Now we can define an equivalence based on net representation of algebraic formulas.

Definition 17. Two formulas $E, E' \in \mathbf{StAFP_0}$ are *net equivalent* in $StAFP_0$, denoted by $E =_{net} E'$, if $Net(E) \simeq Net(E')$, where $\simeq$ is a net isomorphism, i. e., a coincidence of nets up to renaming their places and transitions.

Table 1. The axiom system Θ_{net}

1. Associativity	**3. Distributivity**
1.1 $E\|(F\|G) = (E\|F)\|G$	3.1 $E;(F\|G) = (E_1;F)\|(E_2;G)$, $\Phi_E{=}\Phi_{E_1}{=}\Phi_{E_2},\ \Omega_E{=}\Omega_{E_1}{\cdot}\Omega_{E_2}$
1.2 $E;(F;G) = (E;F);G$	3.2 $(E\|F);G = (E;G_1)\|(F;G_2)$, $\Phi_G{=}\Phi_{G_1}{=}\Phi_{G_2},\ \Omega_G{=}\Omega_{G_1}{\cdot}\Omega_{G_2}$
1.3 $E \bigtriangledown (F \bigtriangledown G) = (E \bigtriangledown F) \bigtriangledown G$	3.3 $E \bigtriangledown (F\|G) = (E_1 \bigtriangledown F)\|(E_2 \bigtriangledown G)$, $\Phi_E{=}\Phi_{E_1}{=}\Phi_{E_2},\ \Omega_E{=}\Omega_{E_1}{\cdot}\Omega_{E_2}$
2. Commutativity	**4. Probability**
2.1 $E\|F = F\|E$	4.1 $E = E_1\|E_2$, $\Phi_E{=}\Phi_{E_1}{=}\Phi_{E_2},\ \Omega_E{=}\Omega_{E_1}{\cdot}\Omega_{E_2}$
2.2 $E \bigtriangledown F = F \bigtriangledown E$	

5.3. Axiomatization

Let $E \in \mathbf{StAFP_0}$. We can easily extract from E the formula $\Phi_E \in \mathbf{AFP_0}$ specifying the non-stochastic process. For this, we replace each activity (a, ω) contained in E by the action a. We call Φ_E the *structure* of E.

We also define an *action probability function* Ω_E from the set of actions contained in activities of E to the probability interval $(0, 1]$. Let $(a, \omega_1), \ldots, (a, \omega_n)$ are *all* activities of E with action a. Then $\Omega_E(a) = \omega_1 \cdots \omega_n$. This naturally corresponds to the idea that probabilities of synchronized actions are multiplied.

Now, in accordance with equivalence $=_{net}$, the axiom system Θ_{net} can be introduced. It is represented in Table 1, where $a \in Act$ and $E, F, G \in \mathbf{StAFP_0}$.

It is easy to check that the axiom system Θ_{net} is sound w.r.t. the equivalence $=_{net}$.

A formula $E \in \mathbf{StAFP_0}$ is a *totally stratified* one iff it has the form $E = E_1\|\cdots\|E_n$, where $n \geq 0$ and each E_i $(1 \leq i \leq n)$ is a *primitive formula*, i. e., does not contain the concurrency operation $\|$.

Theorem 3. *Any formula $E \in \mathbf{StAFP_0}$ can be transformed (with the use of Θ_{net}) into an equivalent (via $=_{net}$) totally stratified one.*

Proof. Similar to that from [8], since we have no difficulties with probabilities. □

Thus, we can always find components $E_1, \ldots E_n$ of a formula E corresponding to concurrently composed subnets. In this case, $E = E_1\|\cdots\|E_n$ and $\Phi_E = \Phi_{E_1}\|\cdots\|\Phi_{E_n},\ \Omega_E = \Omega_{E_1} \cdots \Omega_{E_n}$.

6. Conclusion

In this paper, we have introduced a new class of Stochastic Petri Nets with labeled transitions and a step semantics for transition firing. For this class of nets, we have proposed several equivalence relations and shown that these equivalences preserve interesting aspects of the system behavior. Equivalence relations can be used to compare different systems and to compute a minimal equivalent representation [4] for a given system. The latter aspect is especially interesting for bisimulation equivalences, for which efficient algorithms have been developed to compute the largest bisimulation for a given net. By representing each equivalence class of this relation by a single marking, we obtain a minimal representation at the state transition level. As a result of comparing the equivalences according to the differentiating power, we obtained a lattice of implications. Thus, we provided a new variant of Stochastic Petri Nets with step semantics, and this naturally corresponds to non-interleaving character of the model. In addition, we have demonstrated application of the equivalences to comparing the stationary behavior of DTSPNs. We have also proposed Stochastic Algebra of Finite Processes $StAFP_0$ for specification of Stochastic A-nets (SANs). We have presented a sound axiomatization of the net equivalence (an isomorphism of net representations of formulas). These results can be considered as the main contribution of the paper.

A possible continuation of this work can be an attempt to define other bisimulation equivalences in the interleaving and step semantics. For example, branching bisimulation [23] can be considered, as well as the variants of back-forth equivalences defined in [21, 22]. For these equivalences we cannot use observable state graphs, since we may need information of the lower level. For example, to define branching relations, we should respect occurrences of invisible transitions and states where they conflict with the others. Thus, we cannot just abstract of invisible transitions from the very beginning. To propose the notions of back-forth bisimulations, we need information about the path of events which came to the present state. Hence, it is not enough even to consider paths of transitions which led from the initial marking to the present one, since the same transitions can happen concurrently or sequentially producing the same marking (in non-safe nets). In such a case, we should have something like processes for stochastic nets and collect events belonging to the paths of such processes.

We may also define true concurrent equivalences for stochastic nets, such as the partial word or pomset equivalences [23, 26]. Step semantics proposed in this paper can be considered as the first attempt to investigate true concurrent semantics for stochastic nets.

We could also enrich our algebraic specifications with ability to describe infinite processes such as the recursion operation. But for this purpose our calculus is too restrictive because of synchronization by names. This means that an action cannot depend on equally named one, which is essential for recursion. A possible decision is to use more flexible calculus to be enriched with stochastic features. We consider Petri Box Calculus (PBC) [2] to be the most appropriate candidate.

References

[1] Ajmone-Marsan M., Balbo G., Conte G. A class of generalized stochastic Petri nets for performance evaluation of multiprocessor systems // ACM Transactions on Computer Systems. — 1984. — Vol. 2, No. 2. — P. 93–122.

[2] Best E., Devillers R., Hall J. G. The box calculus: a new causal algebra with multi-label communication // Lect. Notes Comput. Sci. — 1992. — Vol. 609. — P. 21–69.

[3] Buchholz P. Markovian process algebra: composition and equivalence // Proc. 2^{nd} Work. on Process Algebras and Performance Modelling. — Arbeitsberichte des IMMD. — 1994. — Vol. 27. — P. 11–30.

[4] Buchholz P. A notion of equivalence for stochastic Petri nets // Lect. Notes Comput. Sci. — 1995. — Vol. 935. — P. 161–180.

[5] Buchholz P. Iterative decomposition and aggregation of labelled GSPNs // Lect. Notes Comput. Sci. — 1998. — Vol. 1420. — P. 226–245.

[6] Buchholz P. Exact performance equivalence — an equivalence relation for stochastic automata // Theor. Comput. Sci. — 1999. — Vol. 215, No. 1/2. — P. 263–287.

[7] Chiola G., Ajmone-Marsan M., Balbo G., Conte G. Generalized stochastic Petri nets: a definition at the net level and its implications // IEEE Transactions on Software Engineering. — 1993. — Vol 19, No. 2. — P. 89–107.

[8] Cherkasova L. A. Posets with Non-Actions: A Model for Concurrent Nondeterministic Processes. — Sankt Augustin, 1989. — (Arbeitspapiere der GMD / GMD; No. 403).

[9] Christoff I. Testing equivalence and fully abstract models of probabilistic processes // Lect. Notes Comput. Sci. — 1990. — Vol. 458. — P. 128–140.

[10] Cleaveland R., Parrow J., Steffen B. The concurrency workbench: a semantics based tool for the verification of concurrent systems // ACM Transactions on Programming Languages and Systems. — 1993. — Vol. 15, No. 1. — P. 36–72.

[11] Florin G., Natkin S. Les reseaux de Petri stochastiques // Technique et Science Informatique. — 1985. — Vol. 4, No. 1.

[12] van Glabbeek R. J., Smolka S., Steffen B., Tofts C. Reactive, generative and stratified models for probabilistic processes // LICS'90, Proc. of the 5th Annual IEEE Symp. on Logic in Comput. Sci., Philadelphia. — IEEE Comp. Soc. Press, 1990. — P. 130–141.

[13] Hermanns H., Rettelbach M. Syntax, semantics, equivalences and axioms for MTIPP // Proc. 2^{nd} Work. on Process Algebras and Performance Modelling. — Arbeitsberichte des IMMD, 1994. — Vol. 27.

[14] Hillston J. A Compositional Approach for Performance Modelling: Ph.D. thesis. — University of Edinburgh, Dep. of Comput. Sci., 1994.

[15] Jou C. C., Smolka S. Equivalences, congruences and complete axiomatizations for probabilistic processes // Lect. Notes Comput. Sci. — 1990. — Vol. 458. — P. 367–383.

[16] Kotov V. E., Cherkasova L. A. On structural properties of generalized processes // Lect. Notes Comput. Sci. — 1985. — Vol. 188. — P. 288–306.

[17] Kemeny J. G., Snell J. L. Finite Markov Chains. — Springer, 1976.

[18] Larsen K., Skou A. Bisimulation through probabilistic testing // Information and Computation. — 1991. — Vol. 94. — P. 1–28.

[19] Molloy M. Performance analysis using stochastic Petri nets // IEEE Transactions on Software Engineering. — 1982. — Vol. 31, No. 9. — P. 913–917.

[20] Molloy M. Discrete time stochastic Petri nets // IEEE Transactions on Software Engineering. — 1985. — Vol. 11, No. 4. — P. 417–423.

[21] De Nicola R., Montanari U., Vaandrager F. W. Back and forth bisimulations // Proc. CONCUR'90. — Lect. Notes Comput. Sci. — 1990. — Vol. 458. — P. 152–165.

[22] Pinchinat S. Bisimulations for the Semantics of Reactive Systems: Ph.D. Thesis. — Institut National Politechnique de Grenoble, January 1993 (in French).

[23] Pomello L., Rozenberg G., Simone C. A survey of equivalence relations for net based systems // Proc. Advances in Petri Nets 1992. — Lect. Notes Comput. Sci. — 1992. — Vol. 609. —P. 410–472.

[24] Stewart W. J. Introduction to the Numerical Solution of Markov Chains. — Princeton University Press, 1994.

[25] Tarasyuk I. V. Place bisimulation equivalences for design of concurrent and sequential systems // Electronic Notes in Theor. Comput. Sci. — 1998. — Vol. 18. — 16 p.

[26] Vogler W. Modular construction and partial order semantics of Petri nets // Lect. Notes Comput. Sci. — 1992. — Vol. 625.

[27] Zijal R., Ciardo G., Hommel G. Discrete deterministic and stochastic Petri nets // MMB'97, Aktuelle Probleme der Informatik. — VDE Verlag, 1997. — Band 1.

[28] Zijal R., German R. A new approach to discrete time stochastic Petri nets // Lect. Notes in Control and Information Sci. — 1994. — Vol. 199. —P. 198–204.

Part I

Proof of Proposition 1

It is enough to prove it for $\star = s$, since $\star = i$ is a particular case of the previous one with one-element multisets of actions.

Let $\mathcal{R} : N \underline{\leftrightarrow}_s N'$ and $(M_1, M_2) \in \mathcal{R}$. By the definition of step bisimulation, we have $\forall A \in \mathcal{M}(Act)\ \forall \widetilde{\mathcal{L}} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}$

$$M_1 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{L}} \Leftrightarrow M_2 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{L}}.$$

Let $\mathcal{L} = [M_1]_{\mathcal{R}} = [M_2]_{\mathcal{R}}$. Then we can rewrite the above identity as

$$\mathcal{L} \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{L}},$$

since for all markings from the equivalence class $\mathcal{L}$ their probabilities of moving into $\widetilde{\mathcal{L}}$ as a result of occurrence of the multiset of actions A coincide (they are equal to $\mathcal{Q}$).

Let $(A_1 \cdots A_n, \mathcal{P}) \in StepTraces(N)$. Taking into account the previous identity and $\mathcal{R} : N \underline{\leftrightarrow}_s N'$, we have

$$M_{in} \xrightarrow{A_1}\!\!\!\!\rightarrow_{\mathcal{Q}_1} \mathcal{L}_1 \xrightarrow{A_2}\!\!\!\!\rightarrow_{\mathcal{Q}_2} \cdots \xrightarrow{A_n}\!\!\!\!\rightarrow_{\mathcal{Q}_n} \mathcal{L}_n \Leftrightarrow M'_{in} \xrightarrow{A_1}\!\!\!\!\rightarrow_{\mathcal{Q}_1} \mathcal{L}_1 \xrightarrow{A_2}\!\!\!\!\rightarrow_{\mathcal{Q}_2} \cdots \xrightarrow{A_n}\!\!\!\!\rightarrow_{\mathcal{Q}_n} \mathcal{L}_n.$$

Let us also note that, starting from markings of N (N') to some set of markings $\mathcal{L} \subseteq (RS^*(N) \cup RS^*(N'))$, we can reach only markings of the same net, since observable state graphs of two nets do not communicate.

Now we intend to show that the sum of probabilities of all paths going through markings from $\mathcal{L}_1, \ldots, \mathcal{L}_n$ coincides with the product of $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$, which is essentially the probability of the path going through $\mathcal{L}_1, \ldots, \mathcal{L}_n$ in $RG^*(N)/_{\mathcal{R}}$.

Lemma 1. *For DTSPN N and all n $(1 \le n \le |RG^*(N)/_{\mathcal{R}}|)$ the following holds:*

$$\sum_{\{M_1 \in \mathcal{L}_1, \ldots, M_n \in \mathcal{L}_n \mid M_{in} \xrightarrow{A_1}\!\!\!\rightarrow_{\mathcal{P}_1} M_1 \xrightarrow{A_2}\!\!\!\rightarrow_{\mathcal{P}_2} \cdots \xrightarrow{A_n}\!\!\!\rightarrow_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i = \prod_{i=1}^{n} \mathcal{Q}_i.$$

Proof. We will prove it by induction on n.

- $n = 1$. We should prove that

$$\sum_{\{M_1 \in \mathcal{L}_1 | M_{in} \xrightarrow{A_1}\!\!\!\!\rightarrow_{\mathcal{P}_1} M_1\}} \mathcal{P}_1 = \mathcal{Q}_1.$$

This follows from the definition of the transition relation between markings and sets of markings.

- $n \to n+1$. By induction hypothesis, we have the following equality:

$$\sum_{\{M_1 \in \mathcal{L}_1, \ldots, M_n \in \mathcal{L}_n | M_{in} \xrightarrow{A_1}\!\!\!\!\rightarrow_{\mathcal{P}_1} M_1 \xrightarrow{A_2}\!\!\!\!\rightarrow_{\mathcal{P}_2} \cdots \xrightarrow{A_n}\!\!\!\!\rightarrow_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i = \prod_{i=1}^{n} \mathcal{Q}_i.$$

In addition, we have

$$\sum_{\{M_{n+1} \in \mathcal{L}_{n+1} | M_n \xrightarrow{A_{n+1}}\!\!\!\!\rightarrow_{\mathcal{P}_{n+1}} M_{n+1}\}} \mathcal{P}_{n+1} = \mathcal{Q}_{n+1},$$

again by the definition of the transition relation between markings and sets of markings. Let us note that the above sum does not depend on particular $M_n \in \mathcal{L}_n$, i. e., it is the *same for all paths* of $SG^*(N)$ starting at M_{in} and going through $\mathcal{L}_1, \ldots, \mathcal{L}_n$.

As a result of multiplying the left and the right part of the two above equalities, we obtain

$$\left(\sum_{\{M_1 \in \mathcal{L}_1, \ldots, M_n \in \mathcal{L}_n | M_{in} \xrightarrow{A_1}\!\!\!\!\rightarrow_{\mathcal{P}_1} M_1 \xrightarrow{A_2}\!\!\!\!\rightarrow_{\mathcal{P}_2} \cdots \xrightarrow{A_n}\!\!\!\!\rightarrow_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i \right) \cdot$$

$$\sum_{\{M_{n+1} \in \mathcal{L}_{n+1} | M_n \xrightarrow{A_{n+1}}\!\!\!\!\rightarrow_{\mathcal{P}_{n+1}} M_{n+1}\}} \mathcal{P}_{n+1} = \left(\prod_{i=1}^{n} \mathcal{Q}_i \right) \cdot \mathcal{Q}_{n+1}.$$

By distributivity and with the use of the above note on independence of the sum of current probabilities from the concrete marking M_n, we conclude that

$$\sum_{\{M_1 \in \mathcal{L}_1, \ldots, M_{n+1} \in \mathcal{L}_{n+1} | M_{in} \xrightarrow{A_1}\!\!\!\!\rightarrow_{\mathcal{P}_1} M_1 \xrightarrow{A_2}\!\!\!\!\rightarrow_{\mathcal{P}_2} \cdots \xrightarrow{A_{n+1}}\!\!\!\!\rightarrow_{\mathcal{P}_{n+1}} M_{n+1}\}} \prod_{i=1}^{n+1} \mathcal{P}_i = \prod_{i=1}^{n+1} \mathcal{Q}_i.$$

This ends the proof of the lemma. □

Let us note that the result of this lemma can also be applied to N'.

Now we only need to note that summation over *all equivalence classes* is the same as summation over *all markings*, i. e.,

$$\sum_{\{M_1,\dots,M_n|M_{in}\xrightarrow{A_1}_{\mathcal{P}_1}M_1\xrightarrow{A_2}_{\mathcal{P}_2}\cdots\xrightarrow{A_n}_{\mathcal{P}_n}M_n\}}\prod_{i=1}^{n}\mathcal{P}_i =$$

$$\sum_{\{\mathcal{L}_1,\dots,\mathcal{L}_n|M_{in}\xrightarrow{A_1}_{\mathcal{Q}_1}\mathcal{L}_1\xrightarrow{A_2}_{\mathcal{Q}_2}\cdots\xrightarrow{A_n}_{\mathcal{Q}_n}\mathcal{L}_n\}}\prod_{i=1}^{n}\mathcal{Q}_i =$$

$$\sum_{\{\mathcal{L}_1,\dots,\mathcal{L}_n|M'_{in}\xrightarrow{A_1}_{\mathcal{Q}_1}\mathcal{L}_1\xrightarrow{A_2}_{\mathcal{Q}_2}\cdots\xrightarrow{A_n}_{\mathcal{Q}_n}\mathcal{L}_n\}}\prod_{i=1}^{n}\mathcal{Q}_i =$$

$$\sum_{\{M'_1,\dots,M'_n|M'_{in}\xrightarrow{A_1}_{\mathcal{P}'_1}M'_1\xrightarrow{A_2}_{\mathcal{P}'_2}\cdots\xrightarrow{A_n}_{\mathcal{P}'_n}M'_n\}}\prod_{i=1}^{n}\mathcal{P}'_i.$$

Hence, $(A_1\cdots A_n,\mathcal{P})\in StepTraces(N')$, and we obtain $StepTraces(N) \subseteq StepTraces(N')$. The reverse inclusion is proved by symmetry. ☐

Part II

Proof of Proposition 2

As before, it is enough to prove that $StepTraces(N)\subseteq StepTraces(N')$.

Let $\mathcal{R}: N\underline{\leftrightarrow}_{sb}N'$. We prove the inclusion by induction on the length of traces.

- $n = 1$. Since the initial markings are the only markings in their equivalence class, we have $\forall A\in\mathcal{M}(Act)\ \forall\mathcal{L}\in RS^*(N)/_{\mathcal{R}}$

$$M_{in}\xrightarrow{A}_{\mathcal{Q}}\mathcal{L}\ \Leftrightarrow\ M'_{in}\xrightarrow{A}_{\mathcal{Q}}\mathcal{L}.$$

  However, in this case $\mathcal{Q}$ is exactly the probability of observing A in the first step or the probability of the trace A. Furthermore, let $ps^*[A,M]$ be the probability of being at the marking M after observing A from M_{in}. Then $\forall\mathcal{L}\in RS^*(N)/_{\mathcal{R}}$ the following relation holds (see [6]):

$$ps^*[A,\mathcal{L}\cap RS^*(N)] = \sum_{M\in\mathcal{L}\cap RS^*(N)} ps^*[A,M] =$$

$$\sum_{M'\in\mathcal{L}\cap RS^*(N')} ps^*[A,M'] = ps^*[A,\mathcal{L}\cap RS^*(N')].$$

In addition, $ps^*[A,M_1] = ps^*[A,M_2]$ for $M_1, M_2 \in \mathcal{L}\cap RS^*(N)$ and $ps^*[A,M_1'] = ps^*[A,M_2']$ for $M_1', M_2' \in \mathcal{L}\cap RS^*(N')$. So, the equalities hold for any two markings of the same net such that they are from the same equivalence class.

Consequently, we have $ps^*[A,M] = ps^*[A,\mathcal{L}]/|\mathcal{L}\cap RS^*(N)|$ for $M \in RS^*(N)$ and $ps^*[A,M'] = ps^*[A,\mathcal{L}]/|\mathcal{L}\cap RS^*(N')|$ for $M' \in RS^*(N')$.

- $n \to n+1$. Assume that the above relations are proved for all traces of length n. Let $A_1\cdots A_n$ be the trace of length n and let A_{n+1} be the multiset of actions observed in the step $n+1$. The probability of observing A_{n+1} in N equals

$$\sum_{M\in RS^*(N)} ps^*[A_1\cdots A_n, M]\cdot \sum_{\widetilde{M}\in RS^*(N)} PS^*[A_{n+1}, M, \widetilde{M}].$$

Due to equality of probabilities in an equivalence class, this probability can be rewritten as

$$\sum_{\mathcal{L},\mathcal{K}} \frac{ps^*[A_1\cdots A_n,\mathcal{L}\cap RS^*(N)]PS^*[A_{n+1},\mathcal{L}\cap RS^*(N),\mathcal{K}\cap RS^*(N)]}{|\mathcal{L}\cap RS^*(N)|},$$

where the summation ranges over all $\mathcal{L},\mathcal{K} \in (RS^*(N)\cup RS^*(N'))/_{\mathcal{R}}$. By definition, this is equal to

$$\sum_{\mathcal{L},\mathcal{K}} \frac{ps^*[A_1\cdots A_n,\mathcal{L}\cap RS^*(N')]PS^*[A_{n+1},\mathcal{L}\cap RS^*(N'),\mathcal{K}\cap RS^*(N')]}{|\mathcal{L}\cap RS^*(N')|},$$

which is the probability of observing A_{n+1} in N'. The probabilities of being in $M \in \mathcal{K} \in RS^*(N)/_{\mathcal{R}}$ after observing A_{n+1} are computed as

$$ps^*[A_1\cdots A_n, M] = \sum_{\mathcal{L}} \frac{ps^*[A_1\cdots A_n,\mathcal{L}\cap RS^*(N)]}{|\mathcal{L}\cap RS^*(N)|}\cdot$$

$$\frac{PS^*[A_{n+1},\mathcal{L}\cap RS^*(N),\mathcal{K}\cap RS^*(N)]}{|\mathcal{K}\cap RS^*(N)|},$$

which is the same for all $M \in \mathcal{K} \in RS^*(N)/_{\mathcal{R}}$. Since the above relation holds both for N and N', it is easy to show that also

$$ps^*[A_1 \cdots A_n, \mathcal{L} \cap RS^*(N)] = ps^*[A_1 \cdots A_n, \mathcal{L} \cap RS^*(N')]$$

holds for all $\mathcal{L} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}$, which completes the induction step. □

Part III

Proof of Theorem 2

We prove the theorem for backward bisimulation equivalence by induction on the length n of a trace. The proof for forward bisimulation equivalence is similar.

- $n = 1$. The following relations hold for the probability of observing A_1 in the steady state:

$$\sum_{\mathcal{L}} \sum_{\mathcal{K}} \sum_{M \in \mathcal{L} \cap RS^*(N)} ps^*(M) \sum_{\widetilde{M} \in \mathcal{K} \cap RS^*(N)} PS^*[A_1, M, \widetilde{M}] =$$

$$\sum_{\mathcal{L}} ps^*(\mathcal{L}) \sum_{\mathcal{K}} PS^*[A_1, \mathcal{L}, \mathcal{K}] =$$

$$\sum_{\mathcal{L}} \sum_{\mathcal{K}} \sum_{M' \in \mathcal{L} \cap RS^*(N')} ps^*(M') \sum_{\widetilde{M'} \in \mathcal{K} \cap RS^*(N')} PS^*[A_1, M', \widetilde{M'}], \text{ where}$$

$$PS^*[A, \mathcal{L}, \mathcal{K}] = \sum_{M \in \mathcal{L} \cap RS^*(N)} \sum_{\widetilde{M} \in \mathcal{K} \cap RS^*(N)} PS^*[A, M, \widetilde{M}] =$$

$$\sum_{M' \in \mathcal{L} \cap RS^*(N')} \sum_{\widetilde{M'} \in \mathcal{K} \cap RS^*(N')} PS^*[A, M', \widetilde{M'}].$$

- $n \to n + 1$. The proof for $n = 1$ is based on equal probabilities of the equivalence classes and equal probabilities of states inside the equivalence classes. Thus, we only have to prove that the identity holds after observing an arbitrary step. Together with the proof for $n = 1$, this proves the required identity of traces. Both equalities hold after observing a step A if they hold before observing the step, since we have

$$\sum_{\mathcal{K}} \sum_{M \in \mathcal{K} \cap RS^*(N)} ps^*(M) \sum_{\widetilde{M} \in \mathcal{L} \cap RS^*(N)} PS^*[A, M, \widetilde{M}] =$$

$$\sum_{\mathcal{K}} ps^*(\mathcal{K}) \sum_{\mathcal{L}} PS^*[A, \mathcal{L}, \mathcal{K}] =$$

$$\sum_{\mathcal{K}} \sum_{M' \in \mathcal{K} \cap RS^*(N')} ps^*(M') \sum_{\widetilde{M}' \in \mathcal{L} \cap RS^*(N')} PS^*[A, M', \widetilde{M}'],$$

which implies that probabilities of being in equivalence class $\mathcal{L}$ are equal for N and N'.

Let $ps^*_A(M)$ be the probability of being in $M \in \mathcal{L} \cap RS^*(N)$ after observing A starting with probabilities ps^*:

$$ps^*_A(M) = \sum_{\mathcal{K}} \sum_{\widetilde{M} \in \mathcal{K} \cap RS^*(N)} ps^*(\widetilde{M}) PS^*[A, \widetilde{M}, M] =$$

$$\sum_{\mathcal{K}} ps^*(\mathcal{K}) \cdot \frac{PS^*[A, \mathcal{K}, \mathcal{L}]}{|\mathcal{L} \cap RS^*(N)|} =$$

$$\sum_{\mathcal{K}} \sum_{\widetilde{M} \in \mathcal{K} \cap RS^*(N)} ps^*(\widetilde{M}) PS^*[A, \widetilde{M}, \overline{M}] = ps^*_A(\overline{M}),$$

which shows that $\forall M, \overline{M} \in \mathcal{L} \cap RS^*(N) : \ ps^*_A(M) = ps^*_A(\overline{M})$. The equality of probabilities in an equivalence class for states from $RS^*(N')$ can be proved by the symmetric argument. □